SOLUTIONS: ECE 656 Homework (Week 7)

Mark Lundstrom
Purdue University
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1) In Lecture 15, we derived a current equation for a 2D, n-type conductor and wrote it as $J_n = \sigma_S d\left(F_n/q\right)/dx$. Derive the corresponding equation for a p-type semiconductor.

Solution:

$$I = \frac{2q}{h}\int_{-\infty}^{E_V} T(E) M_V(E)\left(f_1 - f_2\right) dE$$

(channels in the valence band are all below $E = E_V$.)

$$f_1 - f_2 \approx \left(-\frac{\partial f_1}{\partial E}\right) qV \qquad T(E) = \frac{\lambda}{\lambda + L} \rightarrow \frac{\lambda}{L}$$

$$I = \left\{\frac{2q}{h}\int_{-\infty}^{E_V} \lambda(E) M_V(E)\left(-\frac{\partial f_0}{\partial E}\right) dE\right\}\frac{qV}{L}$$

$$J_{px} = -I/W \qquad qV = -\Delta F_p$$

$$J_{px} = \left\{\frac{2q}{h}\int_{-\infty}^{E_V} \lambda(E)\left(M_V(E)/W\right)\left(-\frac{\partial f_0}{\partial E}\right) dE\right\}\frac{\Delta F_p}{L}$$

$$J_{px} = \left\{\frac{2q}{h}\int_{-\infty}^{E_V} \lambda(E)\left(M_V(E)/W\right)\left(-\frac{\partial f_0}{\partial E}\right) dE\right\}\frac{dF_p}{dx} = \sigma_{Sp}\frac{dF_p}{dx}$$

$$\boxed{\begin{aligned} J_{px} &= \sigma_{Sp}\frac{d\left(F_p/q\right)}{dx} \\ \sigma_{Sp} &= \frac{2q^2}{h}\int_{-\infty}^{E_V} \lambda(E)\left(M_V(E)/W\right)\left(-\frac{\partial f_0}{\partial E}\right) dE \end{aligned}}$$

2) In Lecture 15, we derived the drift-diffusion equation for a 2D n-type semiconductor with parabolic energy bands. Repeat the derivation for a 3D semiconductor with parabolic energy bands. **Do not** assume Maxwell-Boltzmann statistics.

Solution:

Begin with:

$$J_{nx} = \sigma \frac{dF_n/q}{dx} \qquad \text{(i)}$$

$$n = N_C \mathcal{F}_{1/2}(\eta_F) \qquad \eta_F = (F_n - E_C)/k_B T \qquad N_C = \frac{1}{4}\left(\frac{2m^* k_B T}{\pi \hbar^2}\right)^{3/2}$$

Now find the gradient of the electrochemical potential:

$$\frac{dn}{dx} = N_C \left\{\frac{d}{d\eta_F}\mathcal{F}_{1/2}(\eta_F)\right\}\frac{d\eta_F}{dx} = N_C \mathcal{F}_{-1/2}(\eta_F)\left\{\frac{dF_n}{dx} - \frac{dE_C}{dx}\right\}\frac{1}{k_B T}$$

$$\frac{dF_n}{dx} = \frac{1}{N_C \mathcal{F}_{-1/2}(\eta_F)} k_B T \frac{dn}{dx} + \frac{dE_C}{dx} = \frac{\mathcal{F}_{1/2}(\eta_F)}{N_C \mathcal{F}_{1/2}(\eta_F)\mathcal{F}_{-1/2}(\eta_F)} k_B T \frac{dn}{dx} + \frac{dE_C}{dx}$$

$$\frac{d(F_n/q)}{dx} = \frac{\mathcal{F}_{1/2}(\eta_F)}{\mathcal{F}_{-1/2}(\eta_F)} \frac{k_B T}{q} \frac{1}{n} \frac{dn}{dx} + \frac{d(E_C/q)}{dx} \qquad \text{(ii)}$$

Insert (ii) in (i)

$$J_{nx} = \sigma_n \frac{dF_n/q}{dx} = \sigma_n \left\{\frac{\mathcal{F}_{1/2}(\eta_F)}{\mathcal{F}_{-1/2}(\eta_F)} \frac{k_B T}{q} \frac{1}{n} \frac{dn}{dx} + \mathcal{E}_x\right\}$$

$$J_{nx} = \sigma_n \mathcal{E}_x + \sigma_n \frac{\mathcal{F}_{1/2}(\eta_F)}{\mathcal{F}_{-1/2}(\eta_F)} \frac{k_B T}{q} \frac{1}{n} \frac{dn}{dx} \qquad \text{(iii)}$$

Now write

$$\sigma_n = nq\mu_n$$

and use in (iii) to find:

$$J_{nx} = nq\mu_n\mathcal{E}_x + \frac{\mathcal{F}_{1/2}(\eta_F)}{\mathcal{F}_{-1/2}(\eta_F)} k_B T \mu_n \frac{dn}{dx} \qquad \text{(iv)}$$

Define the diffusion coefficient as

$$D_n = \frac{\mathcal{F}_{1/2}(\eta_F)}{\mathcal{F}_{-1/2}(\eta_F)} \times \frac{k_B T}{q}\mu_n \qquad \text{(v)}$$

For a nondegenerate semiconductor, $\eta_F = (F_n - E_C)/k_B T << 0$ and we find

$$D_n = \frac{k_B T}{q}\mu_n,$$

which is the familiar Einstein relation.

Finally, use (v) in (iv)

$$\boxed{\begin{aligned} J_{nx} &= nq\mu_n\mathcal{E}_x + qD_n\frac{dn}{dx} \\ \frac{D_n}{\mu_n} &= \frac{\mathcal{F}_{1/2}(\eta_F)}{\mathcal{F}_{-1/2}(\eta_F)}\frac{k_B T}{q} \end{aligned}}$$

3) In 1D, we write $R_{1D} = (1/\sigma_{1D})L$, in 2D $R_{2D} = (1/\sigma_{2D})L/W$, and in 3D $R_{3D} = (1/\sigma_{3D})L/A$. Assuming a degenerate conductor, begin with $G_{ball} = \frac{2q^2}{h}M(E_F)$ and develop expressions for the 1D, 2D, and 3D "ballistic conductivities."

Solution:

$$G_{ball}^{1D} = \frac{2q^2}{h}M_{1D}(E_F) \equiv \sigma_{ball}^{1D}\frac{1}{L}$$

$$\sigma_{ball}^{1D} = \frac{2q^2}{h}M_{1D}(E_F)L$$

$$G_{ball}^{2D} = \frac{2q^2}{h}WM_{2D}(E_F) \equiv \sigma_{ball}^{2D}\frac{W}{L}$$

$$\sigma_{ball}^{2D} = \frac{2q^2}{h} M_{2D}\left(E_F\right) L$$

$$G_{ball}^{3D} = \frac{2q^2}{h} A M_{3D}\left(E_F\right) \equiv \sigma_{ball}^{3D} \frac{A}{L}$$

$$\sigma_{ball}^{3D} = \frac{2q^2}{h} M_{3D}\left(E_F\right) L$$

So the results are:

$$\boxed{\begin{aligned} \sigma_{ball}^{1D} &= \frac{2q^2}{h} M_{1D}\left(E_F\right) L \\ \sigma_{ball}^{2D} &= \frac{2q^2}{h} M_{2D}\left(E_F\right) L \\ \sigma_{ball}^{3D} &= \frac{2q^2}{h} M_{3D}\left(E_F\right) L \end{aligned}}$$

To go further, we need to specify $M\left(E_F\right)$. Let's assume parabolic energy bands:

$$M_{\mathrm{1D}}\left(E_F\right) = H\left(E_F - E_C\right)$$
$$M_{\mathrm{2D}}\left(E_F\right) = \frac{\sqrt{2m^*\left(E_F - E_C\right)}}{\pi\hbar} H\left(E_F - E_C\right)$$
$$M_{\mathrm{3D}}\left(E_F\right) = A\frac{m^*}{2\pi\hbar^2}\left(E_F - E_C\right) H\left(E_F - E_C\right)$$

As an exercise, you might now want to derive the ballistic mobilities in 1D, 2D, and 3D.

4) When we write the resistance as $R = R_{\mathrm{ball}}\left(1 + L/\lambda_0\right)$, we assume a constant (energy-independent) mean-free-path. What is the corresponding expression for an energy dependent mean-free-path, $\lambda\left(E\right)$?

Solution:

Begin with the expression for the conductance:

$$G = \frac{2q^2}{h}\int \mathcal{T}\left(E\right) M\left(E\right)\left(-\frac{\partial f_0}{\partial E}\right) dE$$

$$G = \frac{2q^2}{h}\int \frac{\lambda(E)}{\lambda(E)+L} M(E)\left(-\frac{\partial f_0}{\partial E}\right) dE$$

$$R = \left(\frac{h}{2q^2}\right)\frac{1}{\int \frac{\lambda(E)}{\lambda(E)+L} M(E)\left(-\frac{\partial f_0}{\partial E}\right) dE}$$

The corresponding ballistic resistance is:

$$R_{ball} = \left(\frac{h}{2q^2}\right)\frac{1}{\int M(E)\left(-\frac{\partial f_0}{\partial E}\right) dE}$$

So we can re-write the resistance as:

$$R = R_{ball} + \left[\left(\frac{h}{2q^2}\right)\frac{1}{\int \frac{\lambda(E)}{\lambda(E)+L} M(E)\left(-\frac{\partial f_0}{\partial E}\right) dE} - \frac{1}{\int M(E)\left(-\frac{\partial f_0}{\partial E}\right) dE}\right]$$

$$R = R_{ball} + \left[\left\{\left(\frac{h}{2q^2}\right)\frac{1}{\int M(E)\left(-\frac{\partial f_0}{\partial E}\right) dE}\right\} \times \left\{\frac{\int M(E)\left(-\frac{\partial f_0}{\partial E}\right) dE}{\int \frac{\lambda(E)}{\lambda(E)+L} M(E)\left(-\frac{\partial f_0}{\partial E}\right) dE} - 1\right\}\right]$$

$$R = R_{ball} + \left[R_{ball} \times \left\{\frac{\int M(E)\left(-\frac{\partial f_0}{\partial E}\right) dE}{\int \frac{\lambda(E)}{\lambda(E)+L} M(E)\left(-\frac{\partial f_0}{\partial E}\right) dE} - 1\right\}\right]$$

$$R = R_{ball}\left(1 + L\left\{\left\langle\left\langle \frac{1}{\lambda_{app}} \right\rangle\right\rangle - \frac{1}{L}\right\}\right)$$

$$\left\langle\left\langle \frac{1}{\lambda_{app}} \right\rangle\right\rangle \equiv \frac{\int M(E)\left(-\frac{\partial f_0}{\partial E}\right)dE}{\int \frac{\lambda(E)L}{\lambda(E)+L} M(E)\left(-\frac{\partial f_0}{\partial E}\right)dE}$$

The energy dependent apparent mean-free-path is

$$\frac{1}{\lambda_{app}(E)} \equiv \frac{1}{\lambda(E)} + \frac{1}{L}$$

which depends on the sample length, *L*. The average inverse apparent mean free path is:

$$\left\langle\left\langle \frac{1}{\lambda_{app}} \right\rangle\right\rangle \equiv \frac{\int M(E)\left(-\frac{\partial f_0}{\partial E}\right)dE}{\int \lambda_{app}(E) M(E)\left(-\frac{\partial f_0}{\partial E}\right)dE}$$

Summary of final results:

$$\boxed{\begin{aligned} & R = R_{ball}\left(1 + L\left\{\left\langle\left\langle \frac{1}{\lambda_{app}} \right\rangle\right\rangle - \frac{1}{L}\right\}\right) \\ & \left\langle\left\langle \frac{1}{\lambda_{app}} \right\rangle\right\rangle \equiv \frac{\int M(E)\left(-\frac{\partial f_0}{\partial E}\right)dE}{\int \lambda_{app}(E) M(E)\left(-\frac{\partial f_0}{\partial E}\right)dE} \\ & \frac{1}{\lambda_{app}(E)} \equiv \frac{1}{\lambda(E)} + \frac{1}{L} \end{aligned}}$$

Bottom line: Because the apparent mean-free-path and average inverse apparent mean-free-path depend on sample length, *L,* a plot of *R* vs. *L* is not necessarily linear, when there is a strong energy dependence to the mean-free-path. This effects seems to be more important for phonons, where the mean-free-paths very strongly with energy, than for electrons, where the mean-free-paths tend to vary more slowly with energy.