1) For a 3D diffusive resistor, we relate the current density to the electric field by

$$\mathcal{E}_x = \rho_{3D} J_x \text{ V/m},$$

where $\mathcal{E}_x$ is the electric field in V/m and $J_x$ is the current density in A/m$^2$. Write the corresponding equations in 1D and 2D and determine the units of $\rho_{1D}$, $\rho_{2D} = \rho_S$, and $\rho_{3D}$.

**Solution:**

For 1D, there is no current density, just current, so we have:

$$\mathcal{E}_x = \rho_{1D} I_x$$

in terms of units, we can write:

$$\text{V/m} = (\ ?) \text{ A}$$

To make the units match, we must have: $\rho_{1D} = \text{V/(A-m)} = \text{Ohms/m}$

For 2D, the current density is in A/m:

$$\mathcal{E}_x = \rho_{2D} J_x$$

in terms of units, we can write:

$$\text{V/m} = (\ ?) \text{ A/m}$$

To make the units match, we must have: $\rho_{2D} = \text{V/A} = \text{Ohms}$

For 3D, the current density is in A/m$^2$:

$$\mathcal{E}_x = \rho_{3D} J_x$$

in terms of units, we can write:

$$\text{V/m} = (\ ?) \text{ A/m}^2$$

To make the units match, we must have: $\rho_{3D} = \text{V-m/(A)} = \text{Ohms-m}$

| $\rho_{1D}$ | $\Omega - \text{m}^{-1}$ |
| $\rho_{2D}$ | $\Omega$ |
| $\rho_{3D}$ | $\Omega - \text{m}$ |
**ECE 656 Homework (Week 7) Solutions (continued)**

2) The general expression for conductance,

\[ G = \frac{2q^2}{h} \int \mathcal{T}(E) M(E) \left( -\frac{\partial f_0}{\partial E} \right) dE, \]

can be written as

\[ G = \frac{2q^2}{h} \langle \langle \mathcal{T}(E) \rangle \rangle \langle M(E) \rangle. \]

For a 3D resistor in the diffusive limit,

\[ G_{3D} = \frac{2q^2}{h} \langle \langle \lambda(E) \rangle \rangle \langle M(E)/A \rangle \frac{A}{L}. \]

**Derive** the general expressions for \( \langle M(E)/A \rangle \) and for \( \langle \langle \lambda(E) \rangle \rangle \) in terms of their energy-dependent quantities, \( M(E) \) and \( \lambda(E) \). **HINT:** Begin at the ballistic limit and determine \( \langle M(E)/A \rangle \) first.

**Solution:**

Begin in the ballistic limit, where \( \mathcal{T} = 1 \) and we have:

\[ G = \int_{-\infty}^{\infty} \frac{2q^2}{h} M(E) \left( -\frac{\partial f_0}{\partial E} \right) dE = \frac{2q^2}{h} \langle M \rangle \]

\[ \langle M \rangle = \int M(E) \left( -\frac{\partial f_0}{\partial E} \right) dE \]

Note that:

\[ \int_{-\infty}^{\infty} \left( -\frac{\partial f_0}{\partial E} \right) dE = -\int_{-\infty}^{\infty} df_0 = -f_0(\infty) + f_0(-\infty) = -0 + 1 = 1 \]

so we can write:

\[ \langle M \rangle = \frac{\int_{-\infty}^{\infty} M(E) \left( -\frac{\partial f_0}{\partial E} \right) dE}{\int_{-\infty}^{\infty} \left( -\frac{\partial f_0}{\partial E} \right) dE}. \quad (i) \]

Equation (i) looks like an average. We interpret it as the average number of channels in the Fermi window.
ECE 656 Homework (Week 7) Solutions (continued)

Now include the transmission:

\[ G = \frac{2q^2}{h} \int T(E) M(E) \left( -\frac{\partial f_0}{\partial E} \right) dE \]

so we define:

\[ \langle \langle T(E) \rangle \rangle = \frac{\int T(E) M(E) \left( -\frac{\partial f_0}{\partial E} \right) dE}{\int M(E) \left( -\frac{\partial f_0}{\partial E} \right) dE} \]

In the diffusive limit:

\[ \langle \langle \lambda(E) \rangle \rangle = \frac{\int \left( \frac{\lambda(E)}{L} \right) M(E) \left( -\frac{\partial f_0}{\partial E} \right) dE}{\int M(E) \left( -\frac{\partial f_0}{\partial E} \right) dE} \]

Multiply through by \( L \) to find

\[ \langle \langle \lambda(E) \rangle \rangle = \frac{\int \lambda(E) M(E) \left( -\frac{\partial f_0}{\partial E} \right) dE}{\int M(E) \left( -\frac{\partial f_0}{\partial E} \right) dE} \]

Writing (i) and (ii) on a modes per unit area basis, we find:

\[
\begin{align*}
\langle M \rangle &= \frac{\int (M(E)/A) \left( -\frac{\partial f_0}{\partial E} \right) dE}{\int \left( -\frac{\partial f_0}{\partial E} \right) dE} \\
\langle \langle \lambda \rangle \rangle &= \frac{\int \lambda(E)(M(E)/A) \left( -\frac{\partial f_0}{\partial E} \right) dE}{\int (M(E)/A) \left( -\frac{\partial f_0}{\partial E} \right) dE}
\end{align*}
\]
3) According to equ. (6.9) on p. 182 of Advanced Semiconductor Fundamentals, 2nd Ed., (R.F. Pierret, 2003) the mobility of electrons in bulk Si for $N_D = 10^{14}$ cm$^{-3}$ at $T = 300$ K is 1268 cm$^2$/V-s and at $N_D = 10^{20}$ cm$^{-3}$ it is 95 cm$^2$/V-s. Assume an energy independent mean-free-path and determine the mean-free-path, $\lambda_0$, in both cases.

Solution:
The assumption is that we are dealing with a 3D semiconductor. The conductivity is written as:

$$\sigma = \frac{2q^2}{h} \lambda_0 \left\langle M_{3D} / A \right\rangle$$ (i)

$$\left\langle M_{3D} / A \right\rangle = g_V \frac{m^* k_B T}{2 \pi h^2} F_0 \left(\eta_F\right)$$ (ii)

What effective mass do we use in this expression?
Not the density of state effective mass.
Not the conductivity effective mass.
We should use the “distribution of modes” effective mass (See Jeong, et al., J. Appl. Phys., 107, 023707, 2010).

$$g_V m^* = m_{DOM}^* = 2m_t^* + 4 \sqrt{m_t^* m_i^*} = 2.04m_0$$ (iii)

Before we can use (ii), however, we must determine $\eta_F$.

$$n = N_c F_{1/2} \left(\eta_F\right)$$

$N_c = 1.03 \times 10^{19}$ cm$^{-3}$ (Pierret, Adv. Semiconductor Fundamentals, p. 113)

Note that the effective DOS makes use of the density-of-states effective mass:

$$m_{DOS}^* = 6^{2/3} \left( m_t m_i^* \right)^{1/3} = 1.06m_0,$$

which is quite different from the distribution of modes effective mass.

So now we have a procedure:

Given the carrier density, solve for $\eta_F$: $n_0 = N_c F_{1/2} \left(\eta_F\right)$

1) Next, solve for the average number of channels in the Fermi window:

$$\left\langle M_{3D} / A \right\rangle = \frac{m_{DOM}^* k_B T}{2 \pi h^2} F_0 \left(\eta_F\right)$$

3) Then find the conductivity from the given data: $\sigma = n_0 q \mu_n$

4) Finally, solve for the mean-free-path:

$$\lambda_0 = \frac{\sigma}{\left(2q^2 / h\right) \left\langle M_{3D} / A \right\rangle}$$
ECE 656 Homework (Week 7) Solutions (continued)

Case i): \( n_0 = N_D = 10^{14} \text{ cm}^{-3} \) (non-degenerate)

\[
\eta_F = \ln\left( \frac{n_0}{N_C} \right) = -11.5
\]

\[
\left\langle \frac{M_{3D}}{A} \right\rangle = \frac{m^*_D k_B T}{2\pi h^2} F_0 (-11.5) = 1.07 \times 10^8 \text{ cm}^{-2}
\]

\[
\sigma = n_0 q \mu_n = 10^{14} \times 1.6 \times 10^{-19} \times 1268 = 2.03 \times 10^{-2} \text{ S/cm}
\]

\[
\sigma = n_0 q \mu_n = 10^{14} \times 1.6 \times 10^{-19} \times 1268 = 2.03 \times 10^{-2}
\]

\[
\frac{\sigma}{(2q^2/h)} = 263
\]

\[
\lambda_0 = \frac{\sigma}{(2q^2/h)} \left( \frac{1}{\left\langle \frac{M_{3D}}{A} \right\rangle} \right) = 263 \times \frac{1}{1.07 \times 10^8} = 25 \text{ nm}
\]

\( \lambda_0 = 25 \text{ nm} \)

Case ii): \( n_0 = N_D = 10^{20} \text{ cm}^{-3} \) (degenerate)

\[
n_0 = N_C F_{1/2} (\eta_F) \rightarrow 10^{20} = 1.03 \times 10^{19} F_{1/2} (\eta_F) \rightarrow \eta_F = 5.34
\]

\[
\left\langle \frac{M_{3D}}{A} \right\rangle = \frac{m^*_D k_B T}{2\pi h^2} F_0 (5.34) = 1.1 \times 10^{13} F_0 (5.34) \text{ cm}^{-2} = 5.87 \times 10^{13} \text{ cm}^{-2}
\]

\[
\sigma = n_0 q \mu_n = 10^{20} \times 1.6 \times 10^{-19} \times 95 = 1.52 \times 10^{3}
\]

\[
\frac{\sigma}{(2q^2/h)} = 1.97 \times 10^7
\]

\[
\lambda_0 = \frac{\sigma}{(2q^2/h)} \left( \frac{1}{\left\langle \frac{M_{3D}}{A} \right\rangle} \right) = 1.97 \times 10^7 \times \frac{1}{5.87 \times 10^{13}} = 3.4 \text{ nm}
\]

\( \lambda_0 = 3.4 \text{ nm} \)
ECE 656 Homework (Week 7) Solutions (continued)

4) For solving problems involving 2D electrons in parabolic band semiconductors (i.e. as in the channel of a transistor), we can use the result in Lundstrom and Jeong, Appendix, p. 218, eqn. (A.30)

$$\sigma_s = \frac{2q^2}{h} \lambda_0 \sqrt{\frac{2m^*k_BT}{\pi \hbar}} \frac{\Gamma\left(\frac{3}{2}\right)}{F_{-1/2}(\eta_F)}$$

where \( r \) is the characteristic exponent for scattering,

$$\lambda(E) = \lambda_0 \left[ \left( \frac{E - E_C}{k_BT} \right)^r \right].$$

Use this expression and answer the following two questions.

4a) Assume a constant mfp,

$$\lambda(E) = \lambda_0 \quad \text{(i.e. } r = 0)$$

and \textbf{work out} an expression for the 2D mobility in terms of the mfp. Your results should be valid for any level of carrier degeneracy. \textbf{Simplify} your results for \( T = 0K \) and for non-degenerate conditions.

4b) Assume a “power law” mfp describe by

$$\lambda(E - E_C) = \lambda_0 \left[ \left( \frac{E - E_C}{k_BT} \right)^r \right],$$

where “\( r \)” is a characteristic exponent that describes scattering. Repeat problem 4a) for this energy-dependent mfp. \textbf{Note:} for \( T = 0 \) K, only one energy matters, so it is best just to write \( \lambda(E) \) and not use the power law form.

\textbf{Solution:}

4a) \( r = 0 \) energy independent mfp

$$\sigma_s = \frac{2q^2}{h} \lambda_0 \sqrt{\frac{2m^*k_BT}{\pi \hbar}} \frac{\Gamma\left(\frac{3}{2}\right)}{F_{-1/2}(\eta_F)} \equiv n_S q \mu_n \quad (i)$$

recall that \( n_S = \frac{m^*k_BT}{\pi \hbar^2} F_0(\eta_F) \) (valley degeneracy of one is assumed)

$$\mu_n = \frac{1}{n_S} \frac{2q}{h} \lambda_0 \sqrt{\frac{2m^*k_BT}{\pi \hbar}} \frac{\Gamma\left(\frac{3}{2}\right)}{F_{-1/2}(\eta_F)} = \frac{2q}{h} \lambda_0 \sqrt{\frac{2m^*k_BT}{\pi \hbar}} \frac{\Gamma\left(\frac{3}{2}\right)}{F_{-1/2}(\eta_F)} \frac{m^*k_BT}{\pi \hbar^2} F_0(\eta_F)$$
The result is:

\[ \mu_n = \frac{\lambda_0 \nu_F}{(k_B T / q)} \frac{\mathcal{F}_{-1/2}(n_F)}{\mathcal{F}_0(n_F)} \]

For non-degenerate statistics, this simplifies to:

\[ \mu_n = \frac{\lambda_0 \nu_F}{(k_B T / q)} \]

For degenerate statistics, is it best to begin again with:

\[ \sigma_s = \frac{2q^2}{h} M_{2D}(E_F) \lambda(E_F) = nq \mu_n \]

\[ \mu_n = \frac{1}{n_S} \frac{2q}{h} M_{2D}(E_F) \lambda(E_F) = \frac{1}{m^*/(\pi h^2)(E_F - E_C)} \frac{2q}{h} \frac{\sqrt{2m^*(E_F - E_C)}}{\pi h} \lambda(E_F) \]

\[ \mu_n = \frac{1}{(E_F - E_C)/q} \frac{\sqrt{2(E_F - E_C)/m^*}}{\pi} \lambda(E_F) \]

\[ \mu_n = \frac{1}{(E_F - E_C)/q} \left( \frac{2\nu_F/\pi}{2} \right) \lambda_F \quad (\lambda_F = \lambda(E_F)) \]

\[ \mu_n = \frac{\left( \frac{2\nu_F/\pi}{2} \right) \lambda_F}{(E_F - E_C)/q} \]
4b) energy dependent mfp

Use the result in Lundstrom and Jeong, Appendix, p. 218, eqn. (A.30)

\[ \sigma_s = \frac{2q^2}{\hbar} \lambda_0 \sqrt{\frac{2m^* k_B T}{\pi \hbar}} \Gamma(r + 3/2) F_{r-1/2}(\eta_F) \]

Repeating the derivation of part 2a), we find:

\[ \mu_n = \frac{q \lambda_0}{2 \sqrt{m^* k_B T}} \frac{\Gamma(r + 3/2)}{\Gamma(3/2)} \frac{F_{r-1/2}(\eta_F)}{F_0(\eta_F)} \]

For non-degenerate statistics, this simplifies to:

\[ \mu_n = \frac{\lambda_0 \nu_T / 2}{k_B T / q} \frac{\Gamma(r + 3/2)}{\Gamma(3/2)} \frac{F_{r-1/2}(\eta_F)}{F_0(\eta_F)} \]

For degenerate statistics, the answer is that same as in 4a).

\[ \mu_n = \frac{2 \nu_T / \pi}{(E_F - E_C)/q} \frac{\lambda_0}{2} \frac{\Gamma(r + 3/2)}{\Gamma(3/2)} \frac{F_{r-1/2}(\eta_F)}{F_0(\eta_F)} \]

5) Assume an n-channel MOSFET at \( T = 300 \) K with \( n_s = 10^{13} \) cm\(^{-3} \). Assume that only the lowest subband is occupied and compute \( \langle M_{2D} \rangle \), the average number of modes in the Fermi window per micrometer of channel width.

**Solution:**

The first step is to determine the location of the Fermi level.

\[ n_s = N_{2D} F_0(\eta_F) = g_v \frac{m^* k_B T}{\pi \hbar^3} \ln(1 + e^{\eta_v}) \]  
(1)

For the first unprimed subband,

\[ m^* = m_t = 0.19 m_0 \]

\[ g_v = 2 \]
ECE 656 Homework (Week 7) Solutions (continued)

\[ N_{2D} = g_v \frac{m^* k_b T}{\pi \hbar^2} = 2 \frac{0.19 \times 9.11 \times 10^{-31} \times 1.38 \times 10^{-23} \times 300}{3.14 \times (1.055 \times 10^{-34})^2} \]

\[ N_{2D} = 4.1 \times 10^{12} \text{ cm}^{-2} \]

\[ n_s = N_{2D} \ln(1 + e^{\eta_F}) \rightarrow \eta_F = \ln(e^{n_s/N_{2D}} - 1) = \ln(e^{4.1/4.1} - 1) \]

\[ \eta_F = 2.35 \]

From problem 3):

\[ \langle M \rangle = \frac{\int_0^\infty (M(E)/A) (-\frac{\partial f_0}{\partial E}) dE}{\int_0^\infty (-\frac{\partial f_0}{\partial E}) dE} , \]

which can be worked out in 2D to find

\[ \langle M_{2D} \rangle = g_v \frac{\sqrt{2m^* k_b T}}{\pi \hbar} \frac{\sqrt{\pi}}{2} F_{-1/2}(\eta_F) \]

Putting in numbers:

\[ \langle M_{2D} \rangle = 2 \frac{2 \times 0.19 \times 9.11 \times 10^{-31} \times 1.38 \times 10^{-23} \times 300}{3.14 \times 1.055 \times 10^{-34}} \frac{3.14}{2} F_{-1/2}(2.35) \]

\[ \langle M_{2D} \rangle = 2.025 \times 10^8 \times F_{-1/2}(2.35) = 2.025 \times 10^8 \times 1.607 = 3.25 \times 10^8 \text{ m}^{-1} \]

\[ \langle M_{2D} \rangle = 325 \mu\text{m}^{-1} \]

Consider an \( L = 22 \) nm technology with \( W = L = 0.022 \) micrometers. We would have, \( \langle M_{2D} \rangle = 7 \), a fairly small number.

6) In 1D, we express the resistance of a long (diffusive) resistor by \( R_{1D} = \left( \frac{1}{\sigma_{1D}} \right) L \). In 2D, we write \( R_{2D} = \left( \frac{1}{\sigma_{2D}} \right) L/W \) and in 3D \( R_{3D} = \left( \frac{1}{\sigma_{3D}} \right) L/A \). Assuming a degenerate conductor (i.e. \( T = 0 \) K), begin with \( G_{ball} = \frac{2q^2}{h} M \left( E_F \right) \) and develop expressions for the 1D, 2D, and 3D “ballistic conductivities.”
Solution:

\[ G_{\text{ball}}^{1D} = \frac{2q^2}{h} M_{1D}(E_F) \equiv \sigma_{\text{ball}}^{1D} \frac{1}{L} \]

\[ \sigma_{\text{ball}}^{1D} = \frac{2q^2}{h} M_{1D}(E_F) L \]

\[ G_{\text{ball}}^{2D} = \frac{2q^2}{h} WM_{2D}(E_F) \equiv \sigma_{\text{ball}}^{2D} \frac{W}{L} \]

\[ \sigma_{\text{ball}}^{2D} = \frac{2q^2}{h} M_{2D}(E_F) L \]

\[ G_{\text{ball}}^{3D} = \frac{2q^2}{h} AM_{3D}(E_F) \equiv \sigma_{\text{ball}}^{3D} \frac{A}{L} \]

\[ \sigma_{\text{ball}}^{3D} = \frac{2q^2}{h} M_{3D}(E_F) L \]

So the results are:

\[ \sigma_{\text{ball}}^{1D} = \frac{2q^2}{h} M_{1D}(E_F) L \quad (\Omega/\text{m})^{-1} \]

\[ \sigma_{\text{ball}}^{2D} = \frac{2q^2}{h} M_{2D}(E_F) L \quad (\Omega)^{-1} \]

\[ \sigma_{\text{ball}}^{3D} = \frac{2q^2}{h} M_{3D}(E_F) L \quad (\Omega\cdot\text{m})^{-1} \]

To go further, we need to specify \( M(E_F) \). Let’s assume parabolic energy bands:

\[ M_{1D}(E_F) = H(E_F - E_C) \]

\[ M_{2D}(E_F) = \frac{2m^*(E_F - E_C)}{\pi\hbar} H(E_F - E_C) \]

\[ M_{3D}(E_F) = \frac{m^*}{2\pi\hbar^2} (E_F - E_C) H(E_F - E_C) \]

As an exercise, you might now want to derive the ballistic mobilities in 1D, 2D, and 3D.
ECE 656 Homework (Week 7) Solutions (continued)

7) One can derive a near-equilibrium current equation for a 2D, n-type conductor in the diffusive limit and write it as \( J_n = \sigma_s \left( \frac{F_n}{q} \right) / dx \) A/m. Derive the corresponding equation for a p-type semiconductor.

Solution:

\[
I = \frac{2q}{h} \int_{-\infty}^{E_V} T(E) M_v(E) (f_1 - f_2) dE
\]

(the channels in the valence band are all below \( E = E_V \).)

\[
f_1 - f_2 \approx \left( -\frac{\partial f_1}{\partial E} \right) qV \\
T(E) = \frac{\lambda(E)}{\lambda(E) + L} \rightarrow \frac{\lambda(E)}{L} \quad \text{(diffusive transport)}
\]

\[
I = \left\{ \frac{2q}{h} \int_{-\infty}^{E_V} \lambda(E) M_v(E) \left( -\frac{\partial f_0}{\partial E} \right) dE \right\} \frac{qV}{L}
\]

\[
J_{px} = -I/W \quad \text{A/m (current density in 2D)}
\]

\[
qV = -\Delta F_p
\]

\[
J_{px} = \left\{ \frac{2q}{h} \int_{-\infty}^{E_V} \lambda(E) \left( M_v(E) / W \right) \left( -\frac{\partial f_0}{\partial E} \right) dE \right\} \frac{\Delta F_p}{L}
\]

\[
J_{px} = \left\{ \frac{2q}{h} \int_{-\infty}^{E_V} \lambda(E) \left( M_v(E) / W \right) \left( -\frac{\partial f_0}{\partial E} \right) dE \right\} \frac{dF_p}{dx} = \sigma_{sp} \frac{dF_p}{dx}
\]

\[
J_{px} = \sigma_{sp} \frac{d\left( F_p / q \right)}{dx}
\]

\[
\sigma_{sp} = \frac{2q^2}{h} \int_{-\infty}^{E_V} \lambda(E) \left( M_v(E) / W \right) \left( -\frac{\partial f_0}{\partial E} \right) dE
\]
ECE 656 Homework (Week 7) Solutions (continued)

8) Begin with \( J_{nx} = \sigma_n \frac{d}{dx} \left( \frac{F_n}{q} \right) \) and derive the drift-diffusion equation for a 3D n-type semiconductor with parabolic energy bands. Do not assume Maxwell-Boltzmann statistics.

Solution:

Begin with:

\[
J_{nx} = \sigma_n \frac{dF_n}{dx} 
\]  

(i)

\[
n = N_c F_{1/2}(\eta_F) \quad \eta_F = \left( F_n - E_C \right) / k_B T 
\]

\[
N_c = \frac{1}{4} \left( \frac{2m^* k_B T}{\pi \hbar^2} \right)^{3/2} 
\]

Now find the gradient of the electrochemical potential:

\[
\frac{dn}{dx} = N_c \left\{ \frac{d}{d\eta_F} F_{1/2}(\eta_F) \right\} \frac{d\eta_F}{dx} = N_c F_{-1/2}(\eta_F) \left\{ \frac{dF_n}{dx} - \frac{dE_C}{dx} \right\} \frac{1}{k_B T} 
\]

\[
\frac{dF_n}{dx} = \frac{1}{N_c F_{-1/2}(\eta_F)} k_B T \frac{dn}{dx} + \frac{dE_C}{dx} = \frac{F_{1/2}(\eta_F)}{N_c F_{1/2}(\eta_F) F_{-1/2}(\eta_F)} k_B T \frac{dn}{dx} + \frac{dE_C}{dx} 
\]

\[
\frac{d \left( \frac{F_n}{q} \right)}{dx} = \frac{F_{1/2}(\eta_F)}{F_{-1/2}(\eta_F)} k_B T \frac{1}{q} \frac{dn}{dx} + \frac{d \left( \frac{E_C}{q} \right)}{dx} 
\]

(ii)

Insert (ii) in (i)

\[
J_{nx} = \sigma_n \frac{dF_n}{dx} = \sigma_n \left\{ \frac{F_{1/2}(\eta_F)}{F_{-1/2}(\eta_F)} k_B T \frac{1}{q} \frac{dn}{dx} + E_x \right\} 
\]

\[
J_{nx} = \sigma_n E_x + \sigma_n \frac{F_{1/2}(\eta_F)}{F_{-1/2}(\eta_F)} k_B T \frac{1}{q} \frac{dn}{dx} 
\]

(iii)

Now write

\[
\sigma_n = nq \mu_n 
\]

and use in (iii) to find:

\[
J_{nx} = nq \mu_n E_x + \frac{F_{1/2}(\eta_F)}{F_{-1/2}(\eta_F)} k_B T \mu_n \frac{dn}{dx} 
\]

(iv)
Define the diffusion coefficient as
\[ D_n = \frac{F_{1/2}(\eta_F)}{F_{-1/2}(\eta_F)} \times \frac{k_B T}{q} \mu_n \] (v)

Note that for a nondegenerate semiconductor, \( \eta_F = (F_n - E_c)/k_B T \ll 0 \), and we would find
\[ D_n = \frac{k_B T}{q} \mu_n, \]
which is the familiar Einstein relation for non-degenerate semiconductors.

Finally, use (v) in (iv)

9) When we write the resistance as \( R = R_{ball}(1+L/\lambda_0) \), we assume a constant (energy-independent) mean-free-path. What is the corresponding expression for an energy dependent mean-free-path, \( \lambda(E) \)? Is a plot of resistance vs. length of the resistor a straight line?

Solution:

Begin with the expression for the conductance:

\[ G = \frac{2q^2}{\hbar} \int \mathcal{D}(E) M(E) \left( -\frac{\partial f_0}{\partial E} \right) dE \]

\[ G = \frac{2q^2}{\hbar} \int \frac{\lambda(E)}{\lambda(E) + L} M(E) \left( -\frac{\partial f_0}{\partial E} \right) dE \]

\[ R = \left( \frac{\hbar}{2q^2} \right) \frac{1}{\int \frac{\lambda(E)}{\lambda(E) + L} M(E) \left( -\frac{\partial f_0}{\partial E} \right) dE} \]

(A)

The corresponding ballistic resistance is:

\[ R_{ball} = \left( \frac{\hbar}{2q^2} \right) \frac{1}{\int M(E) \left( -\frac{\partial f_0}{\partial E} \right) dE} \]
**ECE 656 Homework (Week 7) Solutions** (continued)

So we can re-write the resistance by adding the ballistic resistance to the RHS and then subtracting it (in the square brackets below):

\[
R = R_{\text{ball}} + \left( \frac{h}{2q^2} \right) \left[ \frac{1}{\int \frac{\lambda(E)}{\lambda(E) + L} M(E) \left( -\frac{\partial f_0}{\partial E} \right) dE} - \frac{1}{\int M(E) \left( -\frac{\partial f_0}{\partial E} \right) dE} \right] \quad (B)
\]

Now multiple and divide by \( \int M(E) \left( -\frac{\partial f_0}{\partial E} \right) dE \) inside the square brackets:

\[
R = R_{\text{ball}} + \left\{ \frac{h}{2q^2} \right\} \frac{1}{\int M(E) \left( -\frac{\partial f_0}{\partial E} \right) dE} \times \left\{ \frac{\int M(E) \left( -\frac{\partial f_0}{\partial E} \right) dE}{\int \frac{\lambda(E)}{\lambda(E) + L} M(E) \left( -\frac{\partial f_0}{\partial E} \right) dE} - 1 \right\}
\]

The first term inside the square brackets is just the ballistic resistance.

\[
R = R_{\text{ball}} \times \left[ \frac{\int M(E) \left( -\frac{\partial f_0}{\partial E} \right) dE}{\int \frac{\lambda(E)}{\lambda(E) + L} M(E) \left( -\frac{\partial f_0}{\partial E} \right) dE} - 1 \right]
\]

\[
R = R_{\text{ball}} \left[ 1 + \frac{\int M(E) \left( -\frac{\partial f_0}{\partial E} \right) dE}{\int \frac{\lambda(E)}{\lambda(E) + L} M(E) \left( -\frac{\partial f_0}{\partial E} \right) dE} \right] - 1 \quad (C)
\]

Now multiply the curly brackets by \( L \) and divide both terms inside the brackets by \( L \).

\[
R = R_{\text{ball}} \left[ 1 + \frac{\int M(E) \left( -\frac{\partial f_0}{\partial E} \right) dE}{\int \frac{\lambda(E)L}{\lambda(E) + L} M(E) \left( -\frac{\partial f_0}{\partial E} \right) dE} - \frac{1}{L} \right] L
\]

which can be written as
$$R = R_{ball} \left[ 1 + \frac{1}{\int \left( \frac{1}{\lambda(E)} + \frac{1}{L} \right)^{-1} M(E) \left(-\frac{\partial f_0}{\partial E}\right) dE - \frac{1}{L} } L \right]$$

**ECE 656 Homework (Week 7) Solutions (continued)**

Now define an “apparent mean-free-path”

$$\frac{1}{\lambda_{app}(E)} = \frac{1}{\lambda(E)} + \frac{1}{L}$$

so we can write the resistance as

$$R = R_{ball} \left[ 1 + \frac{1}{\int \lambda_{app}(E) M(E) \left(-\frac{\partial f_0}{\partial E}\right) dE - \frac{1}{L} } L \right]$$

Note that the apparent mfp is basically the smaller of the actual mfp and the length of the resistor. Now define the average apparent mfp:

$$\left\langle \lambda_{app} \rightangle = \frac{\int \lambda_{app}(E) M(E) \left(-\frac{\partial f_0}{\partial E}\right) dE}{\int M(E) \left(-\frac{\partial f_0}{\partial E}\right) dE}$$

so the resistance vs. length is

$$R = R_{ball} \left[ 1 + \frac{1}{\left\langle \lambda_{app} \rightangle - \frac{1}{L} } L \right]$$

When the mfp is energy dependent, then the average mean-free-path depends on the length of the sample. This occurs because some energy channels and be ballistic and some diffusive and this changes as the sample length changes. Let’s consider the constant mfp case, then we see

$$\left\langle \lambda_{app} \rightangle = \frac{\lambda_0 L}{\lambda_0 + 1}$$

$$\frac{1}{\left\langle \lambda_{app} \rightangle} - \frac{1}{L} = \lambda_0$$

and

$$R = R_{ball} \left[ 1 + \frac{1}{\left\langle \lambda_{app} \rightangle - \frac{1}{L} } L \right] = R_{ball} \left[ 1 + \frac{L}{\lambda_0} \right]$$
The resistance is linear with length, as expected.

**ECE 656 Homework (Week 7) Solutions (continued)**

Summary of final results:

\[
R = R_{\text{ball}} \left[ 1 + \left\{ \frac{1}{\langle \lambda_{\text{app}} \rangle} - \frac{1}{L} \right\} L \right]
\]

\[
\langle \langle \lambda_{\text{app}} \rangle \rangle = \frac{\int \lambda_{\text{app}}(E) M(E) \left( -\frac{\partial f_{0}}{\partial E} \right) dE}{\int M(E) \left( -\frac{\partial f_{0}}{\partial E} \right) dE}
\]

\[
\frac{1}{\lambda_{\text{app}}(E)} \equiv \frac{1}{\lambda(E)} + \frac{1}{L}
\]

**Bottom line:** The algebra is tedious, but the concept is easy to understand. When the mfp is energy dependent, then the transmission of each channel depends differently on energy, depending on its mfp and the sample length. The fraction of channels that are near-ballistic and near-diffusive changes with sample length, so the resistance does not scale linearly with length.

Because the apparent mean-free-path and average inverse apparent mean-free-path depend on sample length, \( L \), a plot of \( R \) vs. \( L \) is not necessarily linear, when there is a strong energy dependence to the mean-free-path. This effect is more important for phonons, where the mean-free-paths very strongly with energy, than for electrons, where the mean-free-paths tend to vary more slowly with energy and for which only a few energy channels near the bottom of the band are typically occupied.