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The BTE with a B-field

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Introduction

Electrical measurements in the presence of a B-field are widely-used for characterizing semiconductors.

Question: How does the current equation change in the presence of a B-field?

Answer: $J_x = \sigma_n \mathcal{E}_x \rightarrow \vec{J} = \left[\sigma\left(\vec{B}\right)\right]\vec{\mathcal{E}} = J_i = \sigma_{ij}\left(\vec{B}\right)\mathcal{E}_j$

The BTE with a B-field…

$$\frac{\partial f}{\partial t} + \vec{\upsilon} \bullet \nabla_r f + \vec{F}_e \bullet \nabla_p f = \left.\frac{df}{dt}\right|_{coll}$$

steady-state with RTA:

$$\vec{\upsilon} \bullet \nabla_r f + \vec{F}_e \bullet \nabla_p f = -\frac{\delta f}{\tau_m}$$

$$\vec{F}_e = -q\vec{\mathcal{E}} - q\vec{\upsilon} \times \vec{B}$$

spatially uniform:

$$-q\vec{\mathcal{E}} \bullet \nabla_p f - q\left(\vec{\upsilon} \times \vec{B}\right) \bullet \nabla_p f = -\delta f / \tau_m$$

$$\nabla_p f \to \nabla_p f_S \ ?$$

The BTE with a B-field…

$$-q\vec{\mathcal{E}}\bullet\nabla_p f_S - q\left(\vec{\upsilon}\times\vec{B}\right)\bullet\nabla_p f_S = -\delta f/\tau_m$$

OK here

But not here!

$$\nabla_p f_S = \frac{\partial f_S}{\partial E}\nabla_p E = \frac{\partial f_S}{\partial E}\vec{\upsilon} \qquad \left(\vec{\upsilon}\times\vec{B}\right)\bullet\vec{\upsilon} = 0\,!$$

$$\boxed{-q\vec{\mathcal{E}}\bullet\nabla_p f_S - q\left(\vec{\upsilon}\times\vec{B}\right)\bullet\nabla_p\left(\delta f\right) = -\delta f/\tau_m}$$

a much more difficult equation to solve

One approach…

$$-q\vec{\mathcal{E}} \bullet \nabla_p f_S - q\left(\vec{\upsilon} \times \vec{B}\right) \bullet \nabla_p f = -\delta f / \tau_m$$

1) First, assume no B-field:

$$-q\vec{\mathcal{E}} \bullet \nabla_p f_S = -\delta f_1 / \tau_m \qquad \delta f_1 = \tau_m \left(-\frac{\partial f_S}{\partial E} \right) \vec{\upsilon} \bullet \left(-q\vec{\mathcal{E}} \right)$$

2) Then use this solution to solve for the additional part due to the B-field:

$$-q\left(\vec{\upsilon} \times \vec{B}\right) \bullet \nabla_p \delta f_1 = -\delta f_2 / \tau_m$$

3) The total solution is:

$$\delta f_{tot} = \delta f_1 + \delta f_2$$

Comments…

$$-q\vec{\mathcal{E}} \bullet \nabla_p f_S - q\left(\vec{\upsilon} \times \vec{B}\right) \bullet \nabla_p \delta f = -\delta f / \tau_m$$

$$\delta f_{tot} = \delta f_1 + \delta f_2$$

1) This approach works and gives us terms ***linear*** in *B*. See Chapter 4, Sec. 4 in Lundstrom, *Fundamentals of Carrier Transport*, 2nd Ed., Cambridge, 2000.

2) Now, we can use δf_{tot} in the BTE to get an even better solution. This gives us additional terms that are ***quadratic*** in *B*.

Another approach…

Recall that without the B-field: $-q\vec{\mathcal{E}} \bullet \nabla_p f_S = -\delta f / \tau_m$

The solution was: $\delta f = \tau_m \left(-\frac{\partial f_0}{\partial E} \right) \vec{\upsilon} \bullet \left(-q\vec{\mathcal{E}} \right)$

So with a B-field: $-q\vec{\mathcal{E}} \bullet \nabla_p f_S - q\left(\vec{\upsilon} \times \vec{B}\right) \bullet \nabla_p \left(\delta f\right) = -\delta f / \tau_m$

Assume a solution of the form: $\delta f = \tau_m \left(-\frac{\partial f_S}{\partial E} \right) \vec{\upsilon} \bullet \vec{G}$

And find the unknown vector, $\vec{G}$

The solution with a B-field…

$$\vec{G} = \frac{-q\vec{\mathcal{E}} - \left(q^2\tau_m / m^*\right)\left(\vec{B}\times\vec{\mathcal{E}}\right) - q\left(q\tau_m / m^*\right)^2\left(\vec{\mathcal{E}} \bullet \vec{B}\right)\vec{B}}{1+\left(\omega_c\tau_m\right)^2}$$

cyclotron frequency: $\omega_c = \dfrac{qB}{m^*}$ low B-field: $\omega_c\tau_m << 1$

(An electron gets only a little way along its orbit, and then it scatters.)

Assume a planar geometry with the electric field in the *x*-*y* plane and a z-directed, **small B-field** normal to the plane.

$$\vec{G} = -q\vec{\mathcal{E}} - \left(q^2\tau_m / m^*\right)\left(\vec{B}\times\vec{\mathcal{E}}\right)$$

G lies in the *x*-*y* plane.

The 2D current equation…

$$\vec{J}_n = \frac{1}{A}\sum_{\vec{k}}(-q)\vec{\upsilon}\left(f_S + \delta f\right) = \frac{1}{A}\sum_{\vec{k}}(-q)\vec{\upsilon}\left(\delta f\right) \qquad \delta f = \tau_m\left(-\frac{\partial f_S}{\partial E}\right)\vec{\upsilon}\bullet\vec{G}$$

Result:

$$\vec{J}_n = \sigma_S\vec{\mathcal{E}} \text{ - } \sigma_S\mu_H\left(\vec{\mathcal{E}}\times\vec{B}\right) \qquad \omega_c\tau_m << 1$$

$$\sigma_S = n_S q\mu_n \qquad \mu_H = \mu_n r_H$$ Hall mobility

$$\mu_n \equiv \frac{q\langle\langle\tau_m\rangle\rangle}{m^*} \qquad r_H \equiv \frac{\langle\langle\tau_m^2\rangle\rangle}{\langle\langle\tau_m\rangle\rangle^2}$$ Hall factor

Average scattering times…

$$\mu_n \equiv \frac{q\langle\langle\tau_m\rangle\rangle}{m^*}$$

$$\langle\langle\tau_m\rangle\rangle \equiv \frac{\langle E\tau(E)\rangle}{\langle E\rangle}$$ "transport average"

$$\langle E\tau(E)\rangle = \frac{\sum_{\vec{k}} E\tau(E) f_S(E)}{\sum_{\vec{k}} f_S(E)} \qquad \langle E\rangle = \frac{\sum_{\vec{k}} E f_S(E)}{\sum_{\vec{k}} f_S(E)}$$

Power law scattering: $\tau_m(E) = \tau_0 (E/k_B T)^s$

$$\langle\langle\tau_m\rangle\rangle = \tau_0 \frac{\Gamma(s+5/2)}{\Gamma(5/2)}$$ (for 3D electrons)

Hall factor

$$r_H \equiv \frac{\langle\langle \tau_m^2 \rangle\rangle}{\langle\langle \tau_m \rangle\rangle^2}$$

$$r_H = \frac{\Gamma(2s+5/2)\Gamma(5/2)}{\Gamma(s+5/2)^2}$$

ADP scattering: *s* = -1/2

II scattering: *s* = 3/2

$$1.18 < r_H < 1.93$$

(for 3D electrons)

Exercise

Assume power law scattering and 2D electrons. Work out the following two quantities in terms of the characteristic exponent, *s*.

$$\langle\langle \tau_m \rangle\rangle \qquad\qquad r_H \equiv \frac{\langle\langle \tau_m^2 \rangle\rangle}{\langle\langle \tau_m \rangle\rangle^2}$$

The magnetoconductivity tensor…

$$\vec{J}_n = \sigma_S \vec{\mathcal{E}} - \sigma_S \mu_H \left(\vec{\mathcal{E}} \times \vec{B}\right)$$ vector notation

$$\begin{pmatrix} J_{nx} \\ J_{ny} \end{pmatrix} = \begin{bmatrix} \sigma_S & -\sigma_S \mu_H B_z \\ +\sigma_S \mu_H B_z & \sigma_S \end{bmatrix} \begin{pmatrix} \mathcal{E}_x \\ \mathcal{E}_y \end{pmatrix}$$

$$J_i = \sigma_{ij}\left(\vec{B}\right)\mathcal{E}_j$$ indicial notation

Diagonal components unaffected.
Off-diagonal components introduced.

Tensors and indicial notation

$$\vec{J} = [\sigma]\vec{\mathcal{E}}$$

$$\begin{bmatrix} J_x \\ J_y \\ J_z \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \begin{bmatrix} \mathcal{E}_x \\ \mathcal{E}_y \\ \mathcal{E}_z \end{bmatrix}$$

$$J_i = \sum_{j=1}^{3} \sigma_{ij} \mathcal{E}_j$$

“indicial notation”

$$J_i = \sum_{j=1}^{j=3} \sigma_{ij} \mathcal{E}_j \equiv \sigma_{ij} \mathcal{E}_j$$

“summation convention”

Diagonal tensors

$$\vec{J} = \sigma_0 \vec{\mathcal{E}}$$

$$\begin{bmatrix} J_x \\ J_y \\ J_z \end{bmatrix} = \begin{bmatrix} \sigma_0 & 0 & 0 \\ 0 & \sigma_0 & 0 \\ 0 & 0 & \sigma_0 \end{bmatrix} \begin{bmatrix} \mathcal{E}_x \\ \mathcal{E}_y \\ \mathcal{E}_z \end{bmatrix}$$

$$J_i = \sigma_0 \mathcal{E}_i$$

“Kronecker delta”

$$\sigma_{ij} = \sigma_0 \delta_{ij}$$

$$\delta_{ij} = 1 \quad (i = j)$$

$$= 0 \quad (i \neq j)$$

$$[\sigma] = \sigma_0 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Cross products

Vector notation: $\vec{J}_n = \sigma_S \vec{\mathcal{E}} - \sigma_S \mu_H \left(\vec{\mathcal{E}} \times \vec{B} \right)$

Indicial notation: ???

Alternating unit tensor

$$\vec{C} = \vec{E} \times \vec{B}$$

$$\vec{C} = \begin{bmatrix} \hat{x} & \hat{y} & \hat{z} \\ \mathcal{E}_x & \mathcal{E}_y & \mathcal{E}_z \\ B_x & B_y & B_z \end{bmatrix} = \hat{x}\left(\mathcal{E}_y B_z - \mathcal{E}_z B_y\right) + \ldots$$

We write cross-products with an "alternating unit tensor":

$$C_i = \varepsilon_{ijk} \mathcal{E}_j B_k$$

$$\begin{aligned} C_x &= \varepsilon_{xjk} \mathcal{E}_j B_k \\ &= \varepsilon_{xyz} \mathcal{E}_y B_z + \varepsilon_{xzy} \mathcal{E}_z B_y \\ &= \mathcal{E}_y B_z - \mathcal{E}_z B_y \end{aligned}$$

$$\begin{aligned} \varepsilon_{ijk} &= +1\left(i, j, k \ \ \text{cyclic}\right) \\ &= -1\left(i, j, k \ \ \text{anti-cyclic}\right) \\ &= 0\left(\text{otherwise}\right) \end{aligned}$$

Current equation

$$\vec{J}_n = \sigma_n \vec{\mathcal{E}} - \left(\sigma_n \mu_n r_H\right) \vec{\mathcal{E}} \times \vec{B}$$

Vector

$$J_i = \sigma_n \mathcal{E}_i - \sigma_n \mu_H \, \varepsilon_{ijk} \mathcal{E}_j B_k$$

Indicial

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See Chapter 4, Sec. 4 in Lundstrom, *Fundamentals of Carrier Transport*, 2nd Ed., Cambridge, 2000.

See Chapter 7 in Lundstrom and Jeong, Near-Equilibrium Transport, World Scientific, 2013.

Questions

$$\vec{J}_n = \sigma_S \vec{\mathcal{E}} - \sigma_S \mu_H \left(\vec{\mathcal{E}} \times \vec{B} \right)$$

$$r_H \equiv \frac{\left\langle\left\langle \tau_m^2 \right\rangle\right\rangle}{\left\langle\left\langle \tau_m \right\rangle\right\rangle^2}$$

$$\tau_m(E) = \tau_0 \left(E/k_B T \right)^s$$

$$r_H = \frac{\Gamma(2s+5/2)\Gamma(5/2)}{\Gamma(s+5/2)^2}$$

(3D)