NAME:________________________________ **PUID:** ________________________________

ECE 656 Exam 1 SOLUTIONS: Fall 2017
September 21, 2017

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This is a closed book exam. You may use a calculator and the formula sheet at the end of this exam.

There are four equally weighted questions. To receive full credit, you must **show your work** (scratch paper is attached).

The exam is designed to be taken in 75 minutes.

Be sure to fill in your name and Purdue student ID at the top of the page.

DO NOT open the exam until told to do so, and stop working immediately when time is called.

The last page is an equation sheet, which you may remove, if you want.

100 points possible, 25 per question

1a)	20 points	2)	25 points	3a)	5 points	4a)	5 points
1b)	5 points			3b)	10 points	4b)	10 points
				3c)	10 points	4c)	10 points

------------------------ Course policy ------------------------

I understand that if I am caught cheating in this exam, I will earn an F for the course and be reported to the Dean of Students.

Read and understood: ________________________________
signature

1) In the next part of the course, we will encounter a quantity:

$$M_{3D}(E) = \frac{m^*}{2\pi\hbar^2}(E - E_C)H(E - E_C),$$

where $H(E-E_C)=0$ for $E<E_C$ and $H(E-E_C)=1$ for $E \geq E_C$

This problem concerns the integral: $I_1 = \int_{E_C}^{\infty} M_{3D}(E) f_0(E)\,dE$, where $f_0(E)$ is the equilibrium Fermi function.

1a) Work out the integral in terms of Fermi-Dirac integrals. **Draw a box around your answer.**

Solution:

$$I_1 = \left(\frac{m^*}{2\pi\hbar^2}\right)\int_{E_C}^{\infty}\frac{(E-E_C)}{1+e^{(E-E_F)/k_BT}}dE$$

define: $\eta = \dfrac{(E-E_C)}{k_BT}$ $\qquad \eta_F = \dfrac{(E_F-E_C)}{k_BT}$

$$I_1 = \left(\frac{m^*}{2\pi\hbar^2}\right)\int_{E_C}^{\infty}\frac{k_BT\eta}{1+e^{(E-E_C+E_C-E_F)/k_BT}}d(k_BT\eta)$$

$$I_1 = \left(\frac{m^*}{2\pi\hbar^2}\right)(k_BT)^2\int_{E_C}^{\infty}\frac{\eta d\eta}{1+e^{\eta-\eta_F}} \qquad \mathcal{F}_1(\eta_F) = \frac{1}{\Gamma(2)}\int_0^{\infty}\frac{\eta d\eta}{1+e^{\eta-\eta_F}} = \int_0^{\infty}\frac{\eta d\eta}{1+e^{\eta-\eta_F}}$$

$$\boxed{I_1 = \left[\frac{m^*(k_BT)^2}{2\pi\hbar^2}\right]\mathcal{F}_1(\eta_F)}$$

1b) Simplify your answer for non-degenerate (Maxwellian) statistics. **Draw a box around your answer.**

Solution:

$\mathcal{F}_1(\eta_F) \to e^{\eta_F}$ for $\eta_F << 0$

$$\boxed{I_1 = \left[\frac{m^*(k_BT)^2}{2\pi\hbar^2}\right]e^{\eta_F} \quad \eta_F << 0}$$

2) This problem concerns an integral slightly different from that in problem 1):

$$I_2 = \int_{E_C}^{\infty} M_{3D}\left(E\right)\left(-\partial f_0/\partial E\right) dE\,.$$

Work out the integral in terms of Fermi-Dirac integrals. **Draw a box around your answer.**

Solution:

$$I_2 = \frac{m^*}{2\pi\hbar^2}\int_{E_C}^{\infty}\left(E-E_C\right)\left(-\partial f_0/\partial E\right) dE$$

$$\left(-\partial f_0/\partial E\right) = +\left(\partial f_0/\partial E_F\right)$$

$$I_2 = \frac{m^*}{2\pi\hbar^2}\frac{\partial}{\partial E_F}\int_{E_C}^{\infty}\left(E-E_C\right) f_0\, dE$$

From prob. 1), we recognize that: $\int_{E_C}^{\infty}\left(E-E_C\right) f_0\, dE = \left(k_B T\right)^2 \mathcal{F}_1\left(\eta_F\right)$

$$I_2 = \frac{m^*}{2\pi\hbar^2}\frac{\partial}{\partial E_F}\left[\left(k_B T\right)^2 \mathcal{F}_1\left(\eta_F\right)\right] = \frac{m^*\left(k_B T\right)^2}{2\pi\hbar^2}\frac{\partial}{\partial E_F}\mathcal{F}_1\left(\eta_F\right)$$

$$I_2 = \frac{m^*\left(k_B T\right)}{2\pi\hbar^2}\frac{\partial}{\partial\left(E_F/k_B T\right)}\mathcal{F}_1\left(\eta_F\right) = \frac{m^*\left(k_B T\right)}{2\pi\hbar^2}\frac{\partial\mathcal{F}_1\left(\eta_F\right)}{\partial\eta_F} = \frac{m^*\left(k_B T\right)}{2\pi\hbar^2}\mathcal{F}_0\left(\eta_F\right)$$

where we have used the property of the Fermi-Dirac integral: $\frac{d\mathcal{F}_j}{d\eta_F} = \mathcal{F}_{j-1}$

Finally,

$$\boxed{I_2 = \left(\frac{m^* k_B T}{2\pi\hbar^2}\right)\mathcal{F}_0\left(\eta_F\right)}$$

3) The definition of the density-of-states for a 2D semiconductor is

$$D_{2D}\left(E\right) \equiv \frac{1}{A}\sum_{\vec{k}_\parallel} \delta\left(E - E_{k_\parallel}\right)$$

where $\vec{k}_\parallel$ is a wavevector in the plane with a magnitude of $k_\parallel = \sqrt{k_x^2 + k_y^2}$. The conduction band of graphene is described by

$$E_{k_\parallel} = \hbar \upsilon_F k_\parallel \qquad E_{k_\parallel} > 0$$

3a) First, convert the sum over $\vec{k}_\parallel$ to an integral in k-space.

Solution:

$$D_{2D}\left(E\right) \equiv \frac{1}{A}\frac{A}{\left(2\pi\right)^2} 2 \int_{k_\parallel=0}^{k_\parallel=\infty} \delta\left(E - E_{k_\parallel}\right) 2\pi k_\parallel dk_\parallel = \frac{1}{\pi}\int_0^\infty \delta\left(E - E_{k_\parallel}\right) k_\parallel dk_\parallel$$

$$\boxed{D_{2D}\left(E\right) = \frac{1}{\pi}\int_0^\infty \delta\left(E - E_{k_\parallel}\right) k_\parallel dk_\parallel}$$

3b) Use the graphene dispersion, $E_{k_\parallel} = \hbar \upsilon_F k_\parallel$, to convert to an integral in energy space.

Solution:

$$k_\parallel = \frac{E_{k_\parallel}}{\hbar \upsilon_F} \qquad dk_\parallel = \frac{dE_{k_\parallel}}{\hbar \upsilon_F} \qquad k_\parallel dk_\parallel = \frac{E_{k_\parallel} dE_{k_\parallel}}{\left(\hbar \upsilon_F\right)^2}$$

$$\boxed{D_{2D}\left(E\right) = \frac{1}{\pi \hbar^2 \upsilon_F^2}\int_0^\infty \delta\left(E - E_{k_\parallel}\right) E_{k_\parallel}\, dE_{k_\parallel}}$$

3c) Evaluate the integral to find $D_{2D}\left(E\right)$ for the conduction band of graphene. Assume now that the valley degeneracy is 2 for graphene.

Solution:

$D_{2D}\left(E\right) = \dfrac{1}{\pi \hbar^2 \upsilon_F^2}\displaystyle\int_0^\infty \delta\left(E - E_{k_\parallel}\right) E_{k_\parallel}\, dE_{k_\parallel} \times 2 = \dfrac{2}{\pi \hbar^2 \upsilon_F^2} E$ a factor of 2 for valley degeneracy has been introduced.

$$\boxed{D_{2D}\left(E\right) = \frac{2E}{\pi \hbar^2 \upsilon_F^2}}$$

4) The material, $In_{0.53}Ga_{0.47}As$ is an important semiconductor because it is lattice matched to InP. It is a direct bandgap material with a bandgap of $E_{G\Gamma} = 0.75$ eV and an effective mass of $m_n^*/m_0 = 0.041$. It contains heavy mass, upper valleys located at an energy of $\Delta E_{\Gamma-L} = 0.55$ eV above the Γ valley minimum. You may assume that $In_{0.53}Ga_{0.47}As$ has a density-of-state effective mass in the upper valleys of $m_{DOS}^*/m_0 = 1$ and that $In_{0.53}Ga_{0.47}As$ has an optical phonon energy of 32 meV. The following two questions concern electron scattering in **undoped** $In_{0.53}Ga_{0.47}As$ at room temperature.

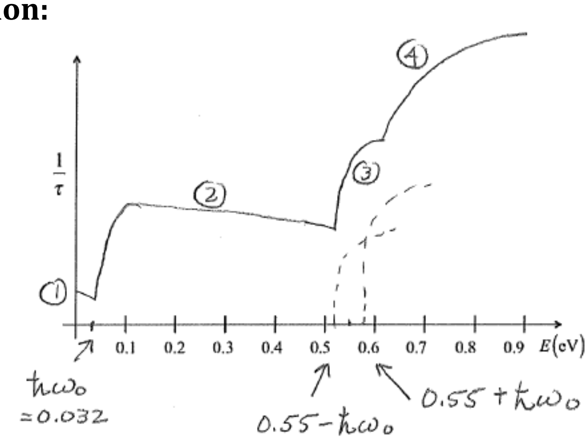
4a) Identify the important phonon scattering mechanisms in undoped $In_{0.53}Ga_{0.47}As$ at room temperature

Solution:
Since it is a polar material, POP scattering should dominate at low energies, as the energy approaches 0.55 eV, we expect inter-valley scattering by phonon absorption and emission to dominate.

Answer: POP and IV

4b) Sketch the total scattering rate vs. energy for **electrons in the Γ valley** from E = 0 (bottom of the Γ valley) to E = 0.9 eV. Your sketch should identify **each** of the main phonon scattering processes and the critical energies.

Solution:



1) POP ABS
2) POP ABS + EMS
3) IV ABS + POP EMS + POP ABS
4) IV ABS + POP EMS + POP ABS + IV EMS

4c) Consider an electron in an L-valley with an energy $E = 0.6$ eV that scatters by optical phonon emission to a state in either the Γ-valley or one of the other L-valleys. Find the ratio of the two scattering rates:

$$\frac{1/\tau_{L\to\Gamma}}{1/\tau_{L\to L}}$$

You may assume that the deformation potentials for the two intervalley processes are identical. **A numerical answer is required.** Note that $E = 0$ is the bottom of the Γ-valley.

Solution:

$$1/\tau_{L\to\Gamma} \propto D_\Gamma\left(E - \hbar\omega_0\right)\times\left(Z_f = 1\right)$$

$$1/\tau_{L\to L} \propto D_L\left(E - 0.55 - \hbar\omega_0\right)\times\left(Z_f = 3\right)$$

$$D_{3D}\left(E\right) = g_v \frac{m^*\sqrt{2m^*\left(E - E_C\right)}}{\pi^2\hbar^3} \propto \left(m^*\right)^{3/2}\sqrt{\left(E - E_C - \hbar\omega_0\right)}$$

$$\frac{1/\tau_{L\to\Gamma}}{1/\tau_{L\to L}} = \left(\frac{0.041}{1}\right)^{3/2}\frac{\sqrt{\left(0.6 - 0.032\right)}}{\sqrt{\left(0.6 - 0.55 - 0.032\right)}}\frac{1}{3} = 0.017$$

$$\boxed{\frac{1/\tau_{L\to\Gamma}}{1/\tau_{L\to L}} \approx 0.02}$$

Electrons in the L-valleys have a 98% probability of scattering to other L-valleys and only a 2% probability of scattering back to the GAMMA valley.

SCRATCH PAPER

ECE-656 Key Equations (Weeks 1-4)

Physical constants:

$\hbar = 1.055\times10^{-34}$ [J-s] $\quad m_0 = 9.109\times10^{-31}$ [kg]

$k_B = 1.380\times10^{-23}$ [J/K] $\quad q = 1.602\times10^{-19}$ [C] $\quad \varepsilon_0 = 8.854\times10^{-14}$ [F/cm]

Density of states in k-space:

1D: $N_k = 2\times(L/2\pi) = L/\pi$ 2D: $N_k = 2\times(A/4\pi^2) = A/2\pi^2$ 3D:
$N_k = 2\times(\Omega/8\pi^2) = \Omega/4\pi^3$

Density of states in energy (parabolic bands, per length, area, or volume):

$$D_{1D}(E) = \frac{g_v}{\pi\hbar}\sqrt{\frac{2m^*}{(E-\varepsilon_1)}} \qquad D_{2D}(E) = g_V\frac{m^*}{\pi\hbar^2} \qquad D_{3D}(E) = g_v\frac{m^*\sqrt{2m^*(E-E_C)}}{\pi^2\hbar^3}$$

Fermi function and Fermi-Dirac Integrals:

$$f_0(E) = \frac{1}{1+e^{(E-E_F)/k_BT}}$$

$$\mathcal{F}_j(\eta_F) = \frac{1}{\Gamma(j+1)}\int_0^\infty \frac{\eta^j d\eta}{1+e^{\eta-\eta_F}} \qquad \mathcal{F}_j(\eta_F) \to e^{\eta} \quad \eta_F << 0 \qquad \frac{d\mathcal{F}_j}{d\eta_F} = \mathcal{F}_{j-1}$$

$\Gamma(n) = (n-1)!$ (n an integer) $\quad \Gamma(1/2) = \sqrt{\pi} \quad \Gamma(p+1) = p\Gamma(p)$

Scattering:

$$S(\vec{p},\vec{p}\,') = \frac{2\pi}{\hbar}\left|H_{\vec{p}\,',\vec{p}}\right|^2\delta(E'-E-\Delta E) \qquad H_{\vec{p}\,',\vec{p}} = \frac{1}{\Omega}\int_{-\infty}^{+\infty} e^{-i\vec{p}\,'\bullet r/\hbar}U_S(\vec{r})e^{i\vec{p}\bullet\vec{r}/\hbar}d\vec{r}$$

$$\frac{1}{\tau(\vec{p})} == \sum_{\vec{p}\,',\uparrow} S(\vec{p},\vec{p}\,') \qquad \frac{1}{\tau_m(\vec{p})} = \sum_{\vec{p}\,',\uparrow} S(\vec{p},\vec{p}\,')\frac{\Delta p_z}{p_{z0}} \qquad \frac{1}{\tau_E(\vec{p})} = \sum_{\vec{p}\,',\uparrow} S(\vec{p},\vec{p}\,')\frac{\Delta E}{E_0}$$

ADP: $\left|K_q\right|^2 = q^2D_A^2$ ODP: $\left|K_q\right|^2 = D_0^2$ PZ: $\left|K_q\right|^2 = (ee_{PZ}/\kappa_S\varepsilon_0)^2$ POP: $\left|K_\beta\right|^2 = \dfrac{\rho e^2\omega_0^2}{q^2\kappa_0\varepsilon_0}\left(\dfrac{\kappa_0}{\kappa_\infty}-1\right)$

$$S(\vec{p},\vec{p}\,') = \frac{\pi}{\Omega\rho\,\omega}\left|K_q\right|^2\left(N_\omega + \frac{1}{2}\mp\frac{1}{2}\right)\delta(\vec{p}\,'-\vec{p}\mp\hbar\vec{q})\delta(E'-E\mp\hbar\omega)$$

$$\delta(\vec{p}\,'-\vec{p}\mp\hbar\vec{q})\delta(E'-E\mp\hbar\omega_\beta) \to \frac{1}{\hbar\upsilon q}\delta\left(\pm\cos\theta + \frac{\hbar q}{2p}\mp\frac{\omega_q}{\upsilon q}\right)$$

$$\frac{1}{\tau} = \frac{1}{\tau_{abs}} + \frac{1}{\tau_{ems}} = \frac{2\pi}{\hbar}\left(\frac{D_A^2k_BT}{c_l}\right)\frac{D_{3D}(E)}{2} \text{ (ADP)} \qquad L_D = \sqrt{\frac{\kappa_S\varepsilon_0k_BT}{q^2n_0}}$$

$$\frac{1}{\tau} = \frac{1}{\tau_m} = \frac{2\pi}{\hbar}\left(\frac{\hbar D_O^2}{2\rho\omega_0}\right)\left(N_0+\frac{1}{2}\mp\frac{1}{2}\right)\frac{D_{3D}(E\pm\hbar\omega_0)}{2} \qquad N_0 = \frac{1}{e^{\hbar\omega_o/k_BT}-1} \text{ (ODP)}$$