NAME:____________________________ **PUID:** ____________________________

ECE 656 Exam 2 SOLUTIONS: Fall 2017

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This is a closed book exam. You may use a calculator and the formula sheet at the end of this exam. **Only the Texas Instruments TI-30X IIS scientific calculator is allowed.**

There are four equally weighted questions. To receive full credit, you must **show your work** (scratch paper is attached).

The exam is designed to be taken in 75 minutes.

Be sure to fill in your name and Purdue student ID at the top of the page.

DO NOT open the exam until told to do so, and stop working immediately when time is called.

The last page is an equation sheet, which you may remove, if you want.

100 points possible, 25 per question

1) 25 points	2a) 10 points	3a) 5 points	4a) 5 points
	2b) 10 points	3b) 10 points	4b) 10 points
	2c) 5 points	3c) 5 points	4c) 10 points
		3d) 5 points	

------------------------ Course policy ------------------------

I understand that if I am caught cheating in this exam, I will earn an F for the course and be reported to the Dean of Students.

Read and understood: ______________________________
signature

1) The mean-free-path for backscattering, λ_0, is key quantity in the Landauer Approach. Assume n-type Ge doped at $N_D = 10^{15}\ \text{cm}^{-3}$. At T = 300 K, the mobility is:

$$\mu_n = 3200\ \text{cm}^2/\text{V-s}$$

Compute the MFP for backscattering, λ_0, at T = 300 K. You may assume that the MFP is independent of energy.

You can make reasonable simplifying assumptions. You will need an effective mass for this calculation. Use the **conductivity effective mass** of n-Ge: $m^* = 0.12 m_0$.

HINT: Begin by determining the diffusion coefficient.

Solution:

$$D_n = \left(k_B T/q\right)\mu_n = 0.026 \times 3200 = 83.2\ \text{cm}^2/\text{s}$$

$$D_n = \frac{\upsilon_T \lambda_0}{2}$$

$$\upsilon_T = \sqrt{2k_B T/\pi m^*} \quad = 1.55 \times 10^7\ \text{cm/s}$$

$$\lambda_0 = \frac{2D_n}{\upsilon_T} = \frac{2(83.2)}{1.55 \times 10^7} = 1.07 \times 10^{-5}\ \text{cm}$$

$$\boxed{\lambda_0 = 107\ \text{nm}}$$

2) Consider a metallic carbon nanotube, a 1D conductor with a linear dispersion, $E(k_x) = \pm\hbar\upsilon_F k_x$. The density of states in this case is a constant, independent of energy, $D(E) = 2g_v/(\pi\hbar\upsilon_F)$, where $g_V = 2$ is the valley degeneracy for a carbon nanotube, and υ_F is the velocity.

2a) Determine the number of channels vs. energy, $M(E)$.

Solution:

$$M(E) \equiv \frac{h}{4}\langle \upsilon_x^+(E)\rangle D_{1D}(E) \qquad \langle \upsilon_x^+(E)\rangle = \upsilon(E)$$

$$M(E) \equiv \frac{h}{4}\upsilon_F \frac{4}{\pi\hbar\upsilon_F} = 2 \qquad \boxed{M(E) = 2}$$

2b) If we put a voltage, V, across a nanotube of length, *L*, a current $I = GV$ will flow. Write down (**but do not evaluate**) an expression for the conductance, *G*. Your answer should be an integral over energy of a quantify that is fully specified. You should assume linear diffusive transport and a MFP for backscattering that is independent of energy.

Solution:

$$G = \frac{2q^2}{h}\int \mathcal{T}(E)M(E)\left(-\frac{\partial f_0}{\partial E}\right)dE \qquad \mathcal{T}(E) = \frac{\lambda(E)}{L} = \frac{\lambda_0}{L} \qquad M(E) = 2$$

$$G = \frac{2q^2}{h}\int\left(\frac{\lambda_0}{L}\right)(2)\left(-\frac{\partial f_0}{\partial E}\right)dE \qquad \boxed{G = \frac{4q^2}{h}\frac{\lambda_0}{L}\int\left(-\frac{\partial f_0}{\partial E}\right)dE}$$

2c) Assume that the Fermi level is in the conduction band ($E > 0$) and evaluate the integral in 2b) to obtain an expression for conductance in terms of the reduced Fermi level, $\eta_F = (E_F - E_C)/k_BT$, where $E_C = 0$.

Solution:

$$G = \frac{4q^2}{h}\frac{\lambda_0}{L}\int\left(-\frac{\partial f_0}{\partial E}\right)dE = \frac{4q^2}{h}\frac{\lambda_0}{L}\frac{\partial}{\partial E_F}\int f_0\, dE \qquad \eta = (E - E_C)/k_BT \qquad \eta_F = (E_F - E_C)/k_BT$$

$$G = \frac{4q^2}{h}\frac{\lambda_0}{L}\frac{1}{k_BT}\frac{\partial}{\partial \eta_F}\int \frac{\eta^0}{1+e^{\eta-\eta_F}}k_BT d\eta$$

$$G = \frac{4q^2}{h}\frac{\lambda_0}{L}\frac{\partial}{\partial \eta_F}\int \frac{\eta^0}{1+e^{\eta-\eta_F}}d\eta = \frac{4q^2}{h}\frac{\lambda_0}{L}\frac{\partial}{\partial \eta_F}\mathcal{F}_0(\eta_F) = \frac{4q^2}{h}\frac{\lambda_0}{L}\mathcal{F}_{-1}(\eta_F)$$

$$\boxed{G = \frac{4q^2}{h}\frac{\lambda_0}{L}\mathcal{F}_{-1}(\eta_F)}$$

3) The Boltzmann Transport Equation is

$$\frac{\partial f}{\partial t} + \vec{\upsilon} \bullet \nabla_r f + \left(-q\vec{\mathcal{E}} - q\vec{\upsilon} \times \vec{B}\right) \bullet \nabla_p f = \left.\frac{df}{dt}\right|_{coll}$$

Answer the following questions.

3a) Simplify the BTE for 1D steady-state, no spatial variations, and no B-fields. Assume the Relaxation Time Approximation.

Solution:

$$\boxed{-q\mathcal{E}_x \bullet \frac{\partial f}{\partial p_x} = -\frac{\delta f}{\tau_m}}$$

3b) Solve the BTE for a metallic carbon nanotube. That is, find $\delta f\left(k_x\right) = f\left(k_x\right) - f_0\left(k_x\right)$. (Your final answer should include a factor, $\left(-\partial f_0 / \partial E\right)$.

Solution:

$$-q\mathcal{E}_x \bullet \frac{\partial f}{\partial p_x} = -\frac{\delta f}{\tau_m} \rightarrow -q\mathcal{E}_x \bullet \frac{\partial f_0}{\partial p_x} = -\frac{\delta f}{\tau_m}$$

$$\delta f = q\tau_m \mathcal{E}_x \bullet \frac{\partial f_0}{\partial p_x} = q\tau_m \mathcal{E}_x \bullet \frac{\partial f_0}{\partial E}\frac{\partial E}{\partial p_x} = q\tau_m \mathcal{E}_x \bullet \frac{\partial f_0}{\partial E}\upsilon_F$$

$$\boxed{\delta f = -q\tau_m \upsilon_F \mathcal{E}_x \left(-\frac{\partial f_0}{\partial E}\right)}$$

3c) From your solution to the BTE, write down an expression for the current in the nanotube. Your final answer should be in terms of an integral in k-space.

Solution:

$I = \frac{1}{L}\sum_{k_x}\left(f_0 + \delta f\right)\left(-q\upsilon_x\right)$ (symmetric part of f contributes nothing, so it can be dropped)

$I = \frac{1}{L}\frac{L}{2\pi}2\int \delta f\left(-q\upsilon_F\right)dk_x = \frac{1}{\pi}\times 2\int_0^{+\infty}\delta f\left(-q\upsilon_F\right)dk_x$ (extra factor of two is to integrate over +kx only)

Now use the results from 3b):

$$I = \frac{2}{\pi}\int_0^{+\infty}\delta f\left(-q\upsilon_F\right)dk_x = \frac{2q^2\upsilon_F^2}{\pi}\mathcal{E}_x\int_0^{+\infty}\tau_m\left(-\frac{\partial f_0}{\partial E}\right)dk_x$$

$$\boxed{I = \frac{2q^2\upsilon_F^2}{\pi}\mathcal{E}_x\int_0^{+\infty}\tau_m\left(-\frac{\partial f_0}{\partial E}\right)dk_x}$$

3d) Evaluate the expression in 3c) to find the current in the nanotube. **Assume an energy-independent scattering time.**

Solution:

$$I = \frac{2q^2\upsilon_F^2\tau_m}{\pi}\mathcal{E}_x\int_0^{+\infty}\left(-\frac{\partial f_0}{\partial E}\right)dk_x = \frac{2q^2\upsilon_F^2\tau_m}{\pi}\mathcal{E}_x\frac{\partial}{\partial E_F}\int_0^{+\infty}f_0\,dk_x$$

convert to integral over energy using the E(k) for the metallic CNT $\frac{dE}{dk_x} = \hbar\upsilon_F$

$$I = \frac{2q^2\upsilon_F^2\tau_m}{\pi}\mathcal{E}_x\frac{\partial}{\partial E_F}\int_0^{+\infty}f_0\frac{dE}{\hbar\upsilon_F} = \frac{2q^2\upsilon_F\tau_m}{\hbar\pi}\mathcal{E}_x\frac{\partial}{\partial E_F}\int_0^{+\infty}f_0\,dE$$

$$I = \frac{4q^2}{h}\left(\upsilon_F\tau_m\right)\mathcal{E}_x\frac{\partial}{k_BT\,\partial\eta_F}\int_0^{+\infty}\frac{k_BT\,d\eta}{1+e^{\eta-\eta_F}} = \frac{4q^2}{h}\left(\upsilon_F\tau_m\right)\mathcal{E}_x\frac{\partial}{\partial\eta_F}\mathcal{F}_0\left(\eta_F\right) = \frac{4q^2}{h}\left(\upsilon_F\tau_m\right)\mathcal{E}_x\mathcal{F}_{-1}\left(\eta_F\right)$$

Now, it is time to account for the valley degeneracy of 2 by multiplying by 2.

$$I = \frac{4q^2}{h}\left(2\upsilon_F\tau_m\right)\mathcal{E}_x\mathcal{F}_{-1}\left(\eta_F\right) = \frac{4q^2}{h}\lambda_0\mathcal{E}_x\mathcal{F}_{-1}\left(\eta_F\right)$$

$\boxed{I = \frac{4q^2}{h}\lambda_0\mathcal{E}_x\mathcal{F}_{-1}\left(\eta_F\right)}$ Since $\mathcal{E}_x$, this is the same answer as in problem 2c)

4) In thermoelectrics, the average energy at which current flows, $E_J = E_C + \Delta_n$ is important. We found that $\Delta_n = 2k_B T$ for a non-degenerate, 3D semiconductor with parabolic energy bands and acoustic deformation potential (ADP) scattering. What is it for a 2D semiconductor with parabolic energy bands and acoustic deformation potential scattering? Answer the following questions.

4a) Assume power law scattering:

$$\lambda(E) = \lambda_0 \left[(E - E_C)/(k_B T) \right]^r$$

What is the characteristic exponent, r, for ADP scattering in a 2D, parabolic band semiconductor?

Solution:

The MFP is proportional to velocity times scattering time.

$$\upsilon \propto (E - E_C)^{1/2}$$

ADP scattering follows the DOS. In 2D, the DOS is independent is energy, so the scattering time is independent of energy. We conclude:

$$\boxed{r = 1/2}$$

4b) The definition of Δ_n is:

$$\Delta_n = \frac{\int (E - E_C)\sigma_n'(E)\,dE}{\int \sigma_n'(E)\,dE}$$

Specify the differential conductivity for this problem (i.e. give an equation for it). You may assume a valley degeneracy of two.

Solution:

$$\sigma'(E) = \frac{2q^2}{h}\lambda(E)\frac{M(E)}{W}\left(-\frac{\partial f_0}{\partial E}\right) \qquad \lambda(E) = \lambda_0 \left[\frac{(E - E_C)}{k_B T} \right]^{1/2} \qquad \frac{M(E)}{W} = g_v \frac{\sqrt{2m^*(E - E_C)}}{\pi\hbar}$$

$$\boxed{\sigma'(E) = \frac{2q^2}{h}\lambda_0 \left[\frac{(E - E_C)}{k_B T} \right]^{1/2} g_v \frac{\sqrt{2m^*(E - E_C)}}{\pi\hbar}\left(-\frac{\partial f_0}{\partial E}\right)}$$

4c) Evaluate the integral in 4b) to determine Δ_n.

Solution:

$$\Delta_n = \frac{\int (E-E_C)\sigma_n'(E)\,dE}{\int \sigma_n'(E)\,dE} = \frac{\text{num}}{\text{den}}$$

The constants cancel out from the numerator and denominator, so:

$$\text{num} = \int_{E_C}^{\infty} \frac{2q^2}{h}\lambda_0 \left[\frac{(E-E_C)}{k_BT}\right]^{1/2} g_v \frac{\sqrt{2m^*(E-E_C)}}{\pi\hbar}\left(-\frac{\partial f_0}{\partial E}\right)dE$$

$$\rightarrow \int_{0}^{\infty} (E-E_C)(E-E_C)^{1/2}(E-E_C)^{1/2}\left(-\frac{\partial f_0}{\partial E}\right)dE$$

$$\text{num} = \int_{E_C}^{\infty} (E-E_C)^2\left(-\frac{\partial f_0}{\partial E}\right)dE = \frac{\partial}{(k_BT)\partial\eta_F}\int_{E_C}^{\infty}(k_BT)^2\eta^2 f_0(k_BT)\,d\eta = (k_BT)^2\frac{\partial}{\partial\eta_F}\int_{E_C}^{\infty}\frac{\eta^2}{1+e^{\eta-\eta_F}}d\eta$$

$$= (k_BT)^2\frac{\partial}{\partial\eta_F}\Gamma(3)\mathcal{F}_2(\eta_F) = 2(k_BT)^2\mathcal{F}_1(\eta_F)$$

The denominator is seen to be:

$$\text{den} = \int_{0}^{\infty}(E-E_C)^{1/2}(E-E_C)^{1/2}\left(-\frac{\partial f_0}{\partial E}\right)dE$$

which can be worked out to find:

$$\text{den} = (k_BT)\frac{\partial}{\partial\eta_F}\Gamma(2)\mathcal{F}_1(\eta_F) = (k_BT)\mathcal{F}_0(\eta_F)$$

We find:

$$\Delta_n = \frac{\text{num}}{\text{den}} = 2k_BT\frac{\mathcal{F}_1(\eta_F)}{\mathcal{F}_0(\eta_F)}$$

In the non-degenerate limit: $\boxed{\Delta_n = 2k_BT}$, which is **exactly the same as 3D with ADP** scattering.

SCRATCH PAPER