Notes on Average Scattering Times and Hall Factors

Jesse Maassen and Mark Lundstrom
Purdue University
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I. Introduction

We write the mobility as
\[
\mu = q \frac{\langle \tau_m \rangle}{m},
\]
and the Hall factor as
\[
\tau_H = \frac{\langle \tau_m^2 \rangle}{\langle \tau_m \rangle^2}.
\]

What are these factors in 1D, 2D, and 3D for parabolic energy bands,
\[
E = E_c + \frac{\hbar^2 k^2}{2m^*},
\]
and power law scattering?
\[
\tau(E - E_c) = \tau_0 \left[\frac{(E - E_c)}{k_B T}\right]^\nu.
\]

II. Solution of the BTE

Begin with the general BTE:
\[
\frac{\partial f}{\partial t} + \vec{v} \cdot \nabla_r f + \vec{F}_e \cdot \nabla_p f = \hat{C} f,
\]
and make the Relaxation Time Approximation to the collision integral,
\[
\hat{C} f = -\left(\frac{f(\vec{p}) - f_0(\vec{p})}{\tau_m}\right) = -\frac{\delta f(\vec{p})}{\tau_m},
\]
to write the steady-state, spatially homogeneous BTE as

\[-q \vec{E} \cdot \nabla_p f = -\frac{\delta f(\vec{p})}{\tau_m}.\]  \hfill (7)

Now assume

\[\nabla_p f \approx \nabla_p f_0\]  \hfill (8)

to find

\[\delta f(\vec{p}) = +q \tau_m \vec{E} \cdot \nabla_p f_0.\]  \hfill (9)

Recognizing that \(f_0\) is a function of energy, we use the chain rule

\[\nabla_p f_0(E) = \left( \frac{\partial f_0}{\partial E} \right) \nabla_p E = \left( \frac{\partial f_0}{\partial E} \right) \vec{v},\]  \hfill (10)

which can be used to write (9) as

\[\delta f(\vec{p}) = \tau_m \left( -\frac{\partial f_0}{\partial E} \right) \vec{v} \cdot (-q \vec{E}).\]  \hfill (11)

Equation (11) is the solution to the steady-state, spatially uniform BTE in the Relaxation Time Approximation. To find the current, we evaluate

\[\vec{J}_n(\vec{r}) = \frac{1}{L^d} \sum_k (-q) \vec{u}(\vec{k}) \delta f(\vec{r}, \vec{k}),\]  \hfill (12)

where \(d = 1, 2,\) or \(3\) depending on the dimension.

For the x-directed current:

\[J_{nx}(\vec{r}) = \frac{1}{L^d} \sum_k (-q) \nu_x(\vec{k}) \delta f(\vec{r}, \vec{k})\]

\[J_{nx}(\vec{r}) = \frac{1}{L^d} \sum_k (-q) \nu_x(\vec{k}) \left\{ \tau_m \left( -\frac{\partial f_0}{\partial E} \right) \vec{v} \cdot (-q \vec{E}) \right\}.\]  \hfill (13)

For an x-directed electric field

\[J_{nx}(\vec{r}) = \left\{ \frac{1}{L^d} \sum_k q^2 \nu_x(\vec{k}) \tau_m \left( -\frac{\partial f_0}{\partial E} \right) \vec{E}_x \right\} = \sigma_{nx} \vec{E}_x,\]

so we have

\[J_{nx} = \sigma_{nx} \vec{E}_x\]
\[ \sigma_{xx} = \frac{1}{L_d} \sum_k q^2 \nu_k^2 (\bar{k}) \tau_m \left( -\frac{\partial f_0}{\partial E} \right). \]  \hfill (14)

Equation (14) is valid in 1D, 2D, or 3D for arbitrary bandstructures.

If we write
\[ \sigma_n \equiv nq \mu_n, \]
then we obtain from (14)
\[ \mu_n = \frac{1}{L_d} \sum_k q^2 \nu_k^2 (\bar{k}) \tau_m \left( -\frac{\partial f_0}{\partial E} \right) = \frac{\sum_k q \nu_k^2 (\bar{k}) \tau_m \left( -\frac{\partial f_0}{\partial E} \right)}{\sum_k f_0}. \]

Now let's assume parabolic energy bands and write the mobility as in (1)
\[ \mu_n = \frac{q \langle \langle \tau_m \rangle \rangle}{m} = \frac{\sum_k q \nu_k^2 \tau_m \left( -\frac{\partial f_0}{\partial E} \right)}{\sum_k f_0} \]
from which we obtain
\[ \langle \langle \tau_m \rangle \rangle = \frac{\sum_k m^* \nu_k^2 \tau_m \left( -\frac{\partial f_0}{\partial E} \right)}{\sum_k f_0}, \]  \hfill (15)
which is the definition of the average scattering time and is valid for parabolic energy bands in 1D, 2D, or 3D.

**Case i): 3D with parabolic energy bands**

Assuming that \( \nu^2 \) is equally distributed between the three degrees of freedom,
\[ \nu_x^2 = \nu_y^2 = \nu_z^2 = \frac{\nu^2}{3}. \]

So (15) becomes
\[ \langle \langle \tau_m \rangle \rangle = \frac{\sum_k m^* \nu_x^2 \tau_m \left( -\frac{\partial f_0}{\partial E} \right)}{3 \sum_k f_0}, \]  \hfill (16)
and for parabolic energy bands
\[ \frac{1}{2} m^* \nu^2 = (E - E_c) \]  \hfill (17)
so (16) becomes
\[
\langle \langle \tau_m \rangle \rangle = \frac{\sum_k (E - E_c) \tau_m \left( - \frac{\partial f_0}{\partial E} \right)}{(3/2) \sum_k f_0}.
\]

Equation (18) is analogous to Eq. (7.76) in Near-Equilibrium Transport (Lundstrom and Jeong).

If we now assume non-degenerate conditions, then
\[
\left( - \frac{\partial f_0}{\partial E} \right) = \frac{1}{k_B T} f_0,
\]
and (18) becomes
\[
\langle \langle \tau_m \rangle \rangle = \frac{\sum_k (E - E_c) \tau_m f_0}{(3k_B T/2) \sum_k f_0}.
\]

Recognizing that the average thermal energy is
\[
W = \frac{3}{2} nk_B T = n \langle (E - E_c) \rangle,
\]
we can express (19) as
\[
\langle \langle \tau_m \rangle \rangle = \frac{\langle (E - E_c) \tau_m \rangle}{\langle (E - E_c) \rangle},
\]
where the average of a quantity, \(X(E)\) is over the equilibrium distribution function according to
\[
\langle X \rangle = \frac{\sum_k X(E) f_0(E)}{\sum_k f_0(E)}.
\]

Equation (20) is valid for a 3D semiconductor with parabolic energy bands under non-degenerate conditions. It is Eq. (3.62) in Fundamentals of Carrier Transport (Lundstrom).

**Case ii): 2D with parabolic energy bands**

Assuming that \(\nu^2\) is equally distributed between the three degrees of freedom,
\[
\nu_x^2 = \nu_y^2 = \frac{\nu^2}{2}.
\]
So (15) becomes
\[ \langle \langle \tau_m \rangle \rangle = \frac{1}{2} \sum_k \rho_k \sum_{f_0} \frac{m^* \nu^2 \tau_m \left( -\frac{\partial f_0}{\partial E} \right)}{f_0}, \quad (22) \]

and for parabolic energy bands, we use (17), so (22) becomes

\[ \langle \langle \tau_m \rangle \rangle = \sum_k \left( E - E_C \right) \tau_m \left( -\frac{\partial f_0}{\partial E} \right) \sum_{f_0} f_0 \sum_{k} \frac{1}{\rho_k \sum_{f_0}}, \quad (23) \]

Equation (23) is Eq. (7.76) in *Near-Equilibrium Transport* (Lundstrom and Jeong).

If we now assume non-degenerate conditions, then

\[ \left( -\frac{\partial f_0}{\partial E} \right) = \frac{1}{k_B T} f_0, \]

and (23) becomes

\[ \langle \langle \tau_m \rangle \rangle = \frac{1}{k_B T} \sum_k \left( E - E_C \right) \tau_m f_0 \sum_{f_0} f_0 \sum_{k} \frac{1}{\rho_k \sum_{f_0}}, \quad (24) \]

Recognizing that the average thermal energy in 2D is

\[ W = nk_B T = n \langle \langle E - E_C \rangle \rangle, \]

we can express (24) as

\[ \langle \langle \tau_m \rangle \rangle = \frac{\langle \langle (E - E_C) \tau_m \rangle \rangle}{\langle \langle (E - E_C) \rangle \rangle}, \quad (25) \]

which is the same as (20).

Equation (25) is valid for a 2D semiconductor with parabolic energy bands under non-degenerate conditions.

**Case iii): 1D with parabolic energy bands**

In 1D,

\[ \nu_x^2 = \nu^2 \]

So (15) becomes

\[ \langle \langle \tau_m \rangle \rangle = \frac{1}{2} \sum_k \frac{m^* \nu^2 \tau_m \left( -\frac{\partial f_0}{\partial E} \right)}{f_0}, \quad (26) \]

and for parabolic energy bands, we use (17), so (26) becomes
\[
\langle \tau_m \rangle = \frac{\sum_k (E - E_C) \tau_m \left( -\frac{\partial f_0}{\partial E} \right)}{(1/2) \sum f_0}.
\]  
(27)

Equation (27) is analogous to Eq. (7.76) in *Near-Equilibrium Transport* (Lundstrom and Jeong).

If we now assume non-degenerate conditions, then

\[
\left( -\frac{\partial f_0}{\partial E} \right) = \frac{1}{k_B T} f_0,
\]

and (27) becomes

\[
\langle \tau_m \rangle = \frac{\sum_k (E - E_C) \tau_m f_0}{(k_B T/2) \sum f_0}.
\]  
(28)

Recognizing that the average thermal energy in 1D is

\[ W = n \frac{k_B T}{2} = n \langle (E - E_C) \rangle, \]

we can express (28) as

\[
\langle \tau_m \rangle = \frac{\langle (E - E_C) \tau_m \rangle}{\langle (E - E_C) \rangle},
\]  
(29)

which is the same as (20) and (25).

Equation (29) is valid for a 1D semiconductor with parabolic energy bands under non-degenerate conditions.

**Summary:**

The general expression for the average scattering time for a semiconductor with parabolic energy bands is

\[
\langle \tau_m \rangle = \frac{\sum_k (E - E_C) \tau_m \left( -\frac{\partial f_0}{\partial E} \right)}{(d/2) \sum f_0},
\]  
(30)

where \( d = 1, 2, \) or \( 3 \) depending on the dimension. For a nondegenerate semiconductor we find (independent of dimension)

\[
\langle \tau_m \rangle = \frac{\langle (E - E_C) \tau_m \rangle}{\langle (E - E_C) \rangle},
\]  
(31)

where the average is over the equilibrium distribution as defined in (21).
III. Exercises: Working out average scattering times

1) For power law scattering in 2D

\[ \langle \tau_m \rangle = \tau_0 \frac{\Gamma(s+2)}{\Gamma(2)}, \]

where \( s \) is the characteristic exponent in the expression

\[ \tau(E - E_C) = \tau_0 \left( \frac{(E - E_C)}{k_B T} \right)^s, \]

and nondegenerate conditions are assumed.

Prove this result.

Solution:

Begin with the definition of average momentum relaxation time:

\[ \langle \tau_m \rangle = \frac{1}{(d/2) \sum_k f_k} \sum_k (E - E_C) \tau_m f_k = \frac{\partial}{\partial \eta} \frac{\sum_k (E - E_C) \tau_m f_k}{\sum_k f_k} \]

\[ \langle \tau_m \rangle = \frac{\text{num}}{\text{denom}} \]

(i)

Consider the numerator first. In 2D, we have:

\[ \text{num} = \frac{1}{k_B T} \frac{\partial}{\partial \eta} \sum_k (E - E_C) \tau_m f_k = \frac{\partial}{\partial \eta} \frac{1}{2\pi} \int_0^\infty \tau_0 \left( \frac{E - E_C}{k_B T} \right)^{s+1} f_0 2\pi k dk \]

(ii)

Assuming parabolic energy bands:

\[ kdk = \frac{m^*}{\hbar^2} dE \]

(iii)

so (ii) becomes

\[ \text{num} = \tau_0 g_v \frac{m^*}{\pi \hbar^2} \frac{\partial}{\partial \eta} \int_0^\infty \left( \frac{E - E_C}{k_B T} \right)^{s+1} f_0 dE \]

(iv)

Change variables to

\[ \eta = \left( \frac{E - E_C}{k_B T} \right) \]

\[ \eta_F = \left( \frac{E_F - E_C}{k_B T} \right) \]

(v)

so

\[ \text{num} = \tau_0 g_v \frac{m^* k_B T}{\pi \hbar^2} \frac{\partial}{\partial \eta} \int_0^\infty \frac{\eta^{s+1}}{1 + e^{\eta - \eta_F}} d\eta \]

(vi)
\[ \text{num} = \tau_0 g_v \frac{m^* k_B T}{\pi h^2} \Gamma(s + 2) \frac{\partial}{\partial \eta_F} F_{s+1}(\eta_F) \]  

(vii)

Now work on the denominator:

\[ \text{denom} = \sum_k f_0 = \frac{1}{2\pi^2} \int_0^\infty f_0 2\pi kdk \]

\[ \text{denom} = \frac{1}{\pi} \int_0^\infty f_0 kdk \]

\[ \text{denom} = g_v \frac{m^*}{\pi h^2} \int_0^\infty f_0 dE \]

\[ \text{denom} = g_v \frac{m^* k_B T}{\pi h^2} \int_0^\infty \frac{d\eta}{1 + e^{\eta - \eta_F}} \]

\[ \text{denom} = g_v \frac{m^* k_B T}{\pi h^2} \Gamma(1) F_0(\eta_F) \]  

(viii)

Using (viii) and (vii), we find

\[ \langle \langle \tau_m \rangle \rangle = \frac{\text{num}}{\text{denom}} = \frac{\tau_0 \Gamma(s + 2) F_s(\eta_F)}{\Gamma(1) F_0(\eta_F)} \]

(ix)

Assuming non-degenerate conditions, we find

\[ \langle \langle \tau_m \rangle \rangle = \tau_0 \frac{\Gamma(s + 2)}{\Gamma(1)} = \tau_0 \frac{\Gamma(s + 2)}{\Gamma(2)} \]

2) Work out the corresponding result in 1D.

**Solution:**

The solution proceeds much as in problem 1).

\[ \langle \langle \tau_m \rangle \rangle = \frac{\sum_k (E - E_C) \tau_m}{(d/2) \sum_k f_0} = \frac{1}{k_B T} \frac{\partial}{\partial \eta_F} \sum_k (E - E_C) \tau_m f_0 = \frac{\text{num}}{\text{denom}} \]
\[
\text{num} = \frac{1}{k_B T} \frac{\partial}{\partial n_f} \sum_k (E - E_C) \tau_m f_0 \\
= \frac{1}{k_B T} \frac{\partial}{\partial n_F} \int_{-\infty}^{\eta} (E - E_C) \tau_0 (\frac{E - E_C}{k_B T}) f_0 dk \\
= 2\tau_0 \frac{\partial}{\partial \eta} \int_0^{\eta} \left( \frac{E - E_C}{k_B T} \right)^{s+1} f_0 dk
\]

Assuming parabolic energy bands:

\[
k = \frac{\sqrt{2m^*(E - E_C)}}{\hbar} \quad dk = \frac{\sqrt{2m^*}}{2h} (E - E_C)^{-1/2} dE
\]

\[
\text{num} = 2\tau_0 \frac{\partial}{\partial \eta} \int_0^{\eta} \left( \frac{E - E_C}{k_B T} \right)^{s+1} f_0 \frac{\sqrt{2m^*}}{2h} (E - E_C)^{-1/2} dE
\]

\[
\text{num} = \tau_0 \frac{\sqrt{2m^*/k_B T}}{\hbar} \frac{\partial}{\partial \eta_F} \int_0^{\eta} \left( \frac{E - E_C}{k_B T} \right)^{s+1/2} f_0 dE
\]

Change variables:

\[
\text{num} = \tau_0 \frac{\sqrt{2m^*/k_B T}}{\hbar} \frac{\partial}{\partial \eta_F} \int_0^{\eta} \frac{\eta^{s+1/2}}{1 + e^{\eta - \eta_F}} d\eta
\]

\[
\text{num} = \tau_0 \frac{\sqrt{2m^*/k_B T}}{\hbar} \Gamma(s + 3/2) \frac{f_{s+1/2}(\eta_F)}{\eta_F}
\]

\[
\text{num} = \tau_0 \frac{\sqrt{2m^*/k_B T}}{\hbar} k_B T \Gamma(s + 3/2) \frac{f_{s+1/2}(\eta_F)}{\eta_F}
\]

\[
\text{denom} = \frac{1}{2} \sum_k f_0 = \frac{1}{2} \int_{-\infty}^{\infty} f_0 dk
\]

\[
\text{denom} = \frac{1}{2} \sum_k f_0 = \int_0^{\infty} f_0 \frac{\sqrt{2m^*}}{2h} (E - E_C)^{-1/2} dE
\]

\[
\text{denom} = \frac{\sqrt{2m^*/k_B T}}{2h} \int_0^{\infty} \left( \frac{E - E_C}{k_B T} \right)^{-1/2} dE
\]

\[
\text{denom} = \frac{\sqrt{2m^*/k_B T}}{2h} \int_0^{\infty} \frac{1 + e^{(E - E_C)/k_B T}}{1 + e^{(E - E_C)/k_B T}} dE
\]
\[
\text{denom} = \frac{\sqrt{2m^* k_B T}}{2h} \int_0^\infty \frac{\eta^{-1/2}}{1 + e^{\eta - \eta_f}} d\eta
\]

\[
\text{denom} = \frac{\sqrt{2m^* k_B T}}{2h} \Gamma(1/2) F_{-1/2}(\eta_F)
\]

Using (x) and (xi)

\[
\langle \langle \tau_m \rangle \rangle = \frac{\text{num}}{\text{denom}} = \tau_0 \frac{\Gamma(s + 3/2) F_{s-1/2}(\eta_F)}{(1/2) \Gamma(1/2) F_{-1/2}(\eta_F)}
\]

\[
\langle \langle \tau_m \rangle \rangle = \frac{\text{num}}{\text{denom}} = \tau_0 \frac{\Gamma(s + 3/2) F_{s-1/2}(\eta_F)}{\Gamma(3/2) F_{-1/2}(\eta_F)}
\]

For non-degenerate conditions power law scattering, and 1D,

\[
\langle \langle \tau_m \rangle \rangle = \tau_0 \frac{\Gamma(s + 3/2)}{\Gamma(3/2)}. \tag{xii}
\]

3) Work out the corresponding result in 3D.

**Solution:**

The solution proceeds much as in problem 1) and 2).

\[
\langle \langle \tau_m \rangle \rangle = \frac{\sum_k (E - E_C) \tau_m \left( \frac{\partial f_0}{\partial E} \right)}{(d/2) \sum_k f_0} = \frac{1}{k_B T \partial \eta_f} \sum_k (E - E_C) \tau_m f_0 \]

\[
\text{num} = \frac{1}{k_B T \partial \eta_f} \sum_k (E - E_C) \tau_m f_0
\]

\[
= \frac{1}{k_B T \partial \eta_f} \int_0^\infty (E - E_C) \tau_m \left( \frac{E - E_C}{k_B T} \right)^{s} f_0 4\pi k^2 \, dk
\]

\[
= 4\pi \tau_0 \frac{\partial}{\partial \eta_f} \int_0^\infty \left( \frac{E - E_C}{k_B T} \right)^{s+1} f_0 k^2 \, dk
\]

Assuming parabolic energy bands:

\[
k^2 = \frac{2m^* (E - E_C)}{\hbar^2} \quad \text{d}k = \frac{\sqrt{2m^*}}{2\hbar} (E - E_C)^{-1/2} \, dE \quad k^2 \, dE = \frac{(2m^*)^{3/2}}{2\hbar^3} (E - E_C)^{1/2} \, dE
\]
num = \(4\pi\tau_0 \frac{\partial}{\partial \eta_F E_c} \left( \frac{E - E_C}{k_B T} \right)^{x+1} f_0 \left(\frac{2m^*}{h^3}\right)^{1/2} \frac{(E - E_C)^{1/2}}{2\hbar^3} dE\)

num = \(\frac{2\pi\tau_0}{h^3} \frac{2m^*}{(2m^*)^{3/2}} \frac{\partial}{\partial \eta_F E_c} \left( \frac{E - E_C}{k_B T} \right)^{x+1} \left(\frac{2m^*}{h^3}\right)^{1/2} f_0 dE\)

num = \(\frac{2\pi\tau_0}{h^3} \frac{2m^*}{(2m^*)^{3/2}} \frac{\sqrt{k_B T}}{\partial \eta_F E_c} \left( \frac{E - E_C}{k_B T} \right)^{x+3/2} f_0 dE\)

Change variables:

num = \(\frac{2\pi\tau_0}{h^3} \frac{2m^*}{(2m^*)^{3/2}} \frac{\partial}{\partial \eta_F E_c} \left( \frac{E - E_C}{k_B T} \right)^{x+3/2} f_0 dE\)

num = \(\frac{2\pi\tau_0}{h^3} \frac{2m^*}{(2m^*)^{3/2}} \frac{\partial}{\partial \eta_F E_c} \Gamma(s + 5/2) F_{s+3/2}(\eta_F)\)

num = \(\frac{2\pi\tau_0}{h^3} \frac{2m^*}{(2m^*)^{3/2}} \Gamma(s + 5/2) F_{s+1/2}(\eta_F)\) (xiii)

Now the denominator:

denom = \(\frac{3}{2} \sum_k f_0 = \frac{3}{2} \int_0^\infty f_0 4\pi k^2 dk\)

denom = \(\frac{3}{2} \int_0^\infty f_0 4\pi k^2 dk = 3\pi \int_0^\infty f_0 \left(\frac{2m^*}{h^3}\right)^{3/2} \left(\frac{E - E_C}{h^3}\right)^{1/2} dE\)

denom = \(\frac{3\pi}{2h^3} \int_0^\infty f_0 (E - E_C)^{1/2} dE\)

denom = \(\frac{3\pi}{2h^3} \Gamma(3/2) F_{1/2}(\eta_F)\) (xiv)

Using (xiii) and (xiv)

\[\langle \tau_m \rangle = \frac{\text{num}}{\text{denom}} = \tau_0 \frac{2\Gamma(s + 5/2)}{3\Gamma(3/2)} \frac{F_{s+1/2}(\eta_F)}{F_{1/2}(\eta_F)} = \tau_0 \frac{\Gamma(s + 5/2)}{3\Gamma(3/2)} \frac{F_{s+1/2}(\eta_F)}{F_{1/2}(\eta_F)}\]

\[\langle \tau_m \rangle = \tau_0 \frac{\Gamma(s + 5/2)}{\Gamma(5/2)} \frac{F_{s+1/2}(\eta_F)}{F_{1/2}(\eta_F)}\] (xv)

For non-degenerate conditions power law scattering, and 3D,

\[\langle \tau_m \rangle = \tau_0 \frac{\Gamma(s + 5/2)}{\Gamma(5/2)}\]
Summary of non-degenerate results:

For power law scattering, parabolic energy bands, and non-degenerate carrier statistics, the "transport average" momentum relaxation time in 1D, 2D, and 3D are:

\[
\begin{align*}
1D : \langle \tau_m \rangle &= \tau_0 \frac{\Gamma(s+3/2)}{\Gamma(3/2)} \\
2D : \langle \tau_m \rangle &= \tau_0 \frac{\Gamma(s+2)}{\Gamma(2)} \\
3D : \langle \tau_m \rangle &= \tau_0 \frac{\Gamma(s+5/2)}{\Gamma(5/2)}
\end{align*}
\]

(32)

Knowing these times, we get the mobility from

\[
\mu = \frac{q \langle \tau_m \rangle}{m^*}
\]

The analogous procedure in the Landauer approach is to relate the mobility to the mean-free-path according to

\[
\mu = \frac{D_n}{k_B T} = \frac{v_T}{2k_B T} \frac{\langle \lambda \rangle}{q}
\]

To compute \( \langle \lambda \rangle \), we assume power law scattering

\[
\lambda(E) = \lambda_0 \left[ \left( E - E_c \right) / k_B T \right]^r
\]

where the characteristic exponent for the mfp is "r" rather than "s". From the definition of the average mean-free-path

\[
\langle \langle \lambda \rangle \rangle = \frac{\int \lambda(E) M(E) \left( -\frac{\partial f_0}{\partial E} \right) dE}{\int M(E) \left( -\frac{\partial f_0}{\partial E} \right) dE}
\]

assuming parabolic bands and nondegenerate conditions, we could express \( \langle \langle \lambda \rangle \rangle \) in terms of Gamma functions.
IV. Hall Factors

The definition of the Hall factor was given in (2) as

\[ r_H = \frac{\langle \tau_m^2 \rangle}{\langle \tau_m \rangle^2}. \]  

(33)

We have worked out the denominator in 1D, 2D, and 3D. For the numerator, we can recognize from (4) that \( \tau_m^2 \) is also in power law form. From (4), we write

\[ \tau_m (E - E_C)^2 = \tau_0^2 \left[ \frac{(E - E_C)}{k_B T} \right]^{2s}, \]  

(34)

so we can evaluate \( \langle \tau_m^2 \rangle \) by using the results for \( \langle \tau_m \rangle \) with \( s \to 2s \).

**Hall Factor in 1D: (non-degenerate)**

\[ \langle \tau_m \rangle = \tau_0 \frac{\Gamma(s + 3/2)}{\Gamma(3/2)} \]

\[ \langle \tau_m^2 \rangle = \tau_0^2 \frac{\Gamma(2s + 3/2)}{\Gamma(3/2)} \]

\[ r_H = \frac{\langle \tau_m^2 \rangle}{\langle \tau_m \rangle^2} = \frac{\tau_0^2 \frac{\Gamma(2s + 3/2)}{\Gamma(3/2)}}{\left[ \frac{\tau_0}{\Gamma(s + 3/2)} \right]^2} = \frac{\Gamma(2s + 3/2) \Gamma(3/2)}{\left[ \Gamma(s + 3/2) \right]^2} \]

**Hall Factor in 2D: (non-degenerate)**

\[ \langle \tau_m \rangle = \tau_0 \frac{\Gamma(s + 2)}{\Gamma(2)} \]

\[ \langle \tau_m^2 \rangle = \tau_0^2 \frac{\Gamma(2s + 2)}{\Gamma(2)} \]
\[ r'_H \left( \frac{\langle \tau^2_m \rangle}{\langle \tau_m \rangle^2} \right) = \frac{\tau_0^2 \frac{\Gamma(2s+2)}{\Gamma(2)}}{\left( \frac{\Gamma(s+2)}{\Gamma(2)} \right)^2} = \frac{\Gamma(2s+2)\Gamma(2)}{\left[ \Gamma(s+2) \right]^2} \]

**Hall Factor in 3D: (non-degenerate)**

\[ \langle \tau^2_m \rangle = \tau_0 \frac{\Gamma(s+5/2)}{\Gamma(5/2)} \]
\[ \langle \tau^2_m \rangle = \tau_0^2 \frac{\Gamma(2s+5/2)}{\Gamma(5/2)} \]
\[ r'_H \left( \frac{\langle \tau^2_m \rangle}{\langle \tau_m \rangle^2} \right) = \frac{\tau_0^2 \frac{\Gamma(2s+5/2)}{\Gamma(5/2)}}{\left( \frac{\Gamma(s+5/2)}{\Gamma(5/2)} \right)^2} = \frac{\Gamma(2s+5/2)\Gamma(5/2)}{\left[ \Gamma(s+5/2) \right]^2} \]

**Summary of Hall factors: (non-degenerate)**

1D: \[ r'_H = \frac{\Gamma(2s+3/2)\Gamma(3/2)}{\left[ \Gamma(s+3/2) \right]^2} \]
2D: \[ r'_H = \frac{\Gamma(2s+2)\Gamma(2)}{\left[ \Gamma(s+2) \right]^2} \]
3D: \[ r'_H = \frac{\Gamma(2s+5/2)\Gamma(5/2)}{\left[ \Gamma(s+5/2) \right]^2} \]