Notes on Average Scattering Times and Hall Factors

Jesse Maassen and Mark Lundstrom
Purdue University
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I. Introduction

We write the mobility as

$$\mu = \frac{q\left\langle\left\langle \tau_m \right\rangle\right\rangle}{m^*} \tag{1}$$

and the Hall factor as

$$r_H = \frac{\left\langle\left\langle \tau_m^2 \right\rangle\right\rangle}{\left\langle\left\langle \tau_m \right\rangle\right\rangle^2} \quad . \tag{2}$$

What are these factors in 1D, 2D, and 3D for parabolic energy bands,

$$E = E_C + \frac{\hbar^2 k^2}{2m^*}, \tag{3}$$

and power law scattering?

$$\tau\left(E - E_C\right) = \tau_0\left[\left(E - E_C\right)/k_B T\right]^s . \tag{4}$$

II. Solution of the BTE

Begin with the general BTE:

$$\frac{\partial f}{\partial t} + \vec{\upsilon} \bullet \nabla_r f + \vec{F}_e \bullet \nabla_p f = \hat{C} f \ , \tag{5}$$

and make the Relaxation Time Approximation to the collision integral,

$$\hat{C} f = -\left(\frac{f(\vec{p}) - f_0(\vec{p})}{\tau_m}\right) = -\frac{\delta f(\vec{p})}{\tau_m}, \tag{6}$$

to write the steady-state, spatially homogeneous BTE as

$$-q\vec{\mathcal{E}} \bullet \nabla_p f = -\frac{\delta f(\vec{p})}{\tau_m}. \tag{7}$$

Now assume

$$\nabla_p f \approx \nabla_p f_0 \tag{8}$$

to find

$$\delta f(\vec{p}) = +q\tau_m \vec{\mathcal{E}} \bullet \nabla_p f_0 . \tag{9}$$

Recognizing that f_0 is a function of energy, we use the chain rule

$$\nabla_p f_0(E) = \left(\frac{\partial f_0}{\partial E}\right)\nabla_p E = \left(\frac{\partial f_0}{\partial E}\right)\vec{\upsilon}, \tag{10}$$

which can be used to write (9) as

$$\delta f(\vec{p}) = \tau_m \left(-\frac{\partial f_0}{\partial E}\right)\vec{\upsilon} \bullet \left(-q\vec{\mathcal{E}}\right). \tag{11}$$

Equation (11) is the solution to the steady-state, spatially uniform BTE in the Relaxation Time Approximation. To find the current, we evaluate

$$\vec{J}_n(\vec{r}) = \frac{1}{L^d}\sum_{\vec{k}}(-q)\vec{\upsilon}(\vec{k})\delta f(\vec{r},\vec{k}), \tag{12}$$

where d = 1, 2, or 3 depending on the dimension.

For the x-directed current:

$$J_{nx}(\vec{r}) = \frac{1}{L^d}\sum_{\vec{k}}(-q)\upsilon_x(\vec{k})\delta f(\vec{r},\vec{k})$$

$$J_{nx}(\vec{r}) = \frac{1}{L^d}\sum_{\vec{k}}(-q)\upsilon_x(\vec{k})\left\{\tau_m\left(-\frac{\partial f_0}{\partial E}\right)\vec{\upsilon}\bullet\left(-q\vec{\mathcal{E}}\right)\right\}. \tag{13}$$

For an x-directed electric field

$$J_{nx}(\vec{r}) = \frac{1}{L^d}\sum_{\vec{k}}q^2\upsilon_x(\vec{k})\left\{\tau_m\left(-\frac{\partial f_0}{\partial E}\right)\upsilon_x\mathcal{E}_x\right\}$$

$$J_{nx}(\vec{r}) = \left\{\frac{1}{L^d}\sum_{\vec{k}}q^2\upsilon_x^2(\vec{k})\tau_m\left(-\frac{\partial f_0}{\partial E}\right)\right\}\mathcal{E}_x = \sigma_n\mathcal{E}_x,$$

so we have

$$J_{nx} = \sigma_{xx}\mathcal{E}_x$$

$$\boxed{\sigma_{xx} = \frac{1}{L^d}\sum_{\vec{k}} q^2 \upsilon_x^2\left(\vec{k}\right)\tau_m\left(-\frac{\partial f_0}{\partial E}\right)}. \tag{14}$$

Equation (14) is valid in 1D, 2D, or 3D for arbitrary bandstructures

If we write

$$\sigma_n \equiv nq\mu_n ,$$

then we obtain from (14)

$$\mu_n = \frac{\frac{1}{L^d}\sum_{\vec{k}} q^2 \upsilon_x^2\left(\vec{k}\right)\tau_m\left(-\frac{\partial f_0}{\partial E}\right)}{nq} = \frac{\sum_{\vec{k}} q\upsilon_x^2\left(\vec{k}\right)\tau_m\left(-\frac{\partial f_0}{\partial E}\right)}{\sum_{\vec{k}} f_0} .$$

Now let's assume parabolic energy bands and write the mobility as in (1)

$$\mu_n = \frac{q\langle\langle\tau_m\rangle\rangle}{m^*} = \frac{\sum_{\vec{k}} q\upsilon_x^2\,\tau_m\left(-\frac{\partial f_0}{\partial E}\right)}{\sum_{\vec{k}} f_0}$$

from which we obtain

$$\langle\langle\tau_m\rangle\rangle = \frac{\sum_{\vec{k}} m^*\upsilon_x^2\,\tau_m\left(-\frac{\partial f_0}{\partial E}\right)}{\sum_{\vec{k}} f_0} , \tag{15}$$

which is the definition of the average scattering time and is valid for parabolic energy bands in 1D, 2D, or 3D.

Case i): 3D with parabolic energy bands

Assuming that υ^2 is equally distributed between the three degrees of freedom,

$$\upsilon_x^2 = \upsilon_y^2 = \upsilon_z^2 = \frac{\upsilon^2}{3} .$$

So (15) becomes

$$\langle\langle\tau_m\rangle\rangle = \frac{\sum_{\vec{k}} \frac{m^*\upsilon^2\tau_m}{3}\left(-\frac{\partial f_0}{\partial E}\right)}{\sum_{\vec{k}} f_0} , \tag{16}$$

and for parabolic energy bands

$$\frac{1}{2}m^*\upsilon^2 = \left(E - E_C\right) \tag{17}$$

so (16) becomes

$$\langle\langle\tau_m\rangle\rangle = \frac{\sum_{\vec{k}}(E-E_C)\tau_m\left(-\frac{\partial f_0}{\partial E}\right)}{(3/2)\sum_{\vec{k}} f_0}. \tag{18}$$

Equation (18) is analogous to Eq. (7.76) in *Near-Equilibrium Transport* (Lundstrom and Jeong).

If we now assume non-degenerate conditions, then

$$\left(-\frac{\partial f_0}{\partial E}\right) = \frac{1}{k_B T} f_0,$$

and (18) becomes

$$\langle\langle\tau_m\rangle\rangle = \frac{\sum_{\vec{k}}(E-E_C)\tau_m f_0}{(3k_B T/2)\sum_{\vec{k}} f_0}. \tag{19}$$

Recognizing that the average thermal energy is

$$W = \frac{3}{2} n k_B T = n\langle(E-E_C)\rangle,$$

we can express (19) as

$$\langle\langle\tau_m\rangle\rangle = \frac{\langle(E-E_C)\tau_m\rangle}{\langle(E-E_C)\rangle}, \tag{20}$$

where the average of a quantity, $X(E)$ is over the equilibrium distribution function according to

$$\langle X\rangle = \frac{\sum_k X(E) f_0(E)}{\sum_k f_0(E)}. \tag{21}$$

Equation (20) is valid for a 3D semiconductor with parabolic energy bands under non-degenerate conditions. It is Eq. (3.62) in *Fundamentals of Carrier Transport* (Lundstrom).

Case ii): 2D with parabolic energy bands

Assuming that υ^2 is equally distributed between the three degrees of freedom,

$$\upsilon_x^2 = \upsilon_y^2 = \frac{\upsilon^2}{2}.$$

So (15) becomes

$$\langle\langle \tau_m \rangle\rangle = \frac{\sum_{\vec{k}} \frac{m^* \upsilon^2 \tau_m}{2} \left(-\frac{\partial f_0}{\partial E}\right)}{\sum_{\vec{k}} f_0}, \tag{22}$$

and for parabolic energy bands, we use (17), so (22) becomes

$$\langle\langle \tau_m \rangle\rangle = \frac{\sum_{\vec{k}} (E - E_C) \tau_m \left(-\frac{\partial f_0}{\partial E}\right)}{\sum_{\vec{k}} f_0}. \tag{23}$$

Equation (23) is Eq. (7.76) in *Near-Equilibrium Transport* (Lundstrom and Jeong).

If we now assume non-degenerate conditions, then

$$\left(-\frac{\partial f_0}{\partial E}\right) = \frac{1}{k_B T} f_0,$$

and (23) becomes

$$\langle\langle \tau_m \rangle\rangle = \frac{\sum_{\vec{k}} (E - E_C) \tau_m f_0}{(k_B T) \sum_{\vec{k}} f_0}. \tag{24}$$

Recognizing that the average thermal energy in 2D is

$$W = n k_B T = n \langle (E - E_C) \rangle,$$

we can express (24) as

$$\langle\langle \tau_m \rangle\rangle = \frac{\langle (E - E_C) \tau_m \rangle}{\langle (E - E_C) \rangle}, \tag{25}$$

which is the same as (20).

Equation (25) is valid for a 2D semiconductor with parabolic energy bands under non-degenerate conditions.

Case iii): 1D with parabolic energy bands

In 1D,

$$\upsilon_x^2 = \upsilon^2$$

So (15) becomes

$$\langle\langle \tau_m \rangle\rangle = \frac{\sum_{\vec{k}} m^* \upsilon^2 \tau_m \left(-\frac{\partial f_0}{\partial E}\right)}{\sum_{\vec{k}} f_0}, \tag{26}$$

and for parabolic energy bands, we use (17), so (26) becomes

$$\langle\langle \tau_m \rangle\rangle = \frac{\sum_{\vec{k}} (E - E_C)\tau_m \left(-\frac{\partial f_0}{\partial E}\right)}{(1/2)\sum_{\vec{k}} f_0}. \tag{27}$$

Equation (27) is analogous to Eq. (7.76) in *Near-Equilibrium Transport* (Lundstrom and Jeong).

If we now assume non-degenerate conditions, then

$$\left(-\frac{\partial f_0}{\partial E}\right) = \frac{1}{k_B T} f_0,$$

and (27) becomes

$$\langle\langle \tau_m \rangle\rangle = \frac{\sum_{\vec{k}} (E - E_C)\tau_m f_0}{(k_B T/2)\sum_{\vec{k}} f_0}. \tag{28}$$

Recognizing that the average thermal energy in 1D is

$$W = n\frac{k_B T}{2} = n\langle (E - E_C) \rangle,$$

we can express (28) as

$$\langle\langle \tau_m \rangle\rangle = \frac{\langle (E - E_C)\tau_m \rangle}{\langle (E - E_C) \rangle}, \tag{29}$$

which is the same as (20) and (25).

Equation (29) is valid for a 1D semiconductor with parabolic energy bands under non-degenerate conditions.

Summary:

The general expression for the average scattering time for a semiconductor with parabolic energy bands is

$$\boxed{\langle\langle \tau_m \rangle\rangle = \frac{\sum_{\vec{k}} (E - E_C)\tau_m \left(-\frac{\partial f_0}{\partial E}\right)}{(d/2)\sum_{\vec{k}} f_0}}, \tag{30}$$

where d = 1, 2, or 3 depending on the dimension. For a nondegenerate semiconductor we find (independent of dimension)

$$\boxed{\langle\langle \tau_m \rangle\rangle = \frac{\langle (E - E_C)\tau_m \rangle}{\langle (E - E_C) \rangle}}, \tag{31}$$

where the average is over the equilibrium distribution as defined in (21).

III. Exercises: Working out average scattering times

1) For power law scattering in 2D

$$\left\langle\left\langle \tau_m \right\rangle\right\rangle = \tau_0 \frac{\Gamma(s+2)}{\Gamma(2)},$$

where *s* is the characteristic exponent in the expression

$$\tau(E-E_C) = \tau_0\left[(E-E_C)/k_BT\right]^s ,$$

and nondegenerate conditions are assumed.

Prove this result.

Solution:

Begin with the definition of average momentum relaxation time:

$$\left\langle\left\langle \tau_m \right\rangle\right\rangle = \frac{\sum_{\vec{k}}(E-E_C)\tau_m\left(-\frac{\partial f_0}{\partial E}\right)}{(d/2)\sum_{\vec{k}} f_0} = \left\langle\left\langle \tau_m \right\rangle\right\rangle = \frac{\frac{\partial}{k_BT\,\partial\eta_F}\sum_{\vec{k}}(E-E_C)\tau_m f_0}{\sum_{\vec{k}} f_0}$$

$$\left\langle\left\langle \tau_m \right\rangle\right\rangle = \frac{\text{num}}{\text{denom}} \qquad \text{(i)}$$

Consider the numerator first. In 2D, we have:

$$\text{num} = \frac{1}{k_BT}\frac{\partial}{\partial\eta_F}\sum_{\vec{k}}(E-E_C)\tau_m f_0 = \frac{\partial}{\partial\eta_F}\frac{1}{2\pi^2}\int_0^\infty \tau_0\left(\frac{E-E_C}{k_BT}\right)^{s+1} f_0 2\pi k dk \qquad \text{(ii)}$$

Assuming parabolic energy bands:

$$kdk = g_V\frac{m^*}{\hbar^2}dE \qquad \text{(iii)}$$

so (ii) becomes

$$\text{num} = \tau_0 g_V\frac{m^*}{\pi\hbar^2}\frac{\partial}{\partial\eta_F}\int_0^\infty\left(\frac{E-E_C}{k_BT}\right)^{s+1} f_0 dE \qquad \text{(iv)}$$

Change variables to

$$\eta = \left(\frac{E-E_C}{k_BT}\right) \quad \eta_F = \left(\frac{E_F-E_C}{k_BT}\right) \qquad \text{(v)}$$

so

$$\text{num} = \tau_0 g_V\frac{m^* k_BT}{\pi\hbar^2}\frac{\partial}{\partial\eta_F}\int_0^\infty\frac{\eta^{s+1}}{1+e^{\eta-\eta_F}}d\eta \qquad \text{(vi)}$$

$$\text{num} = \tau_0 g_V \frac{m^* k_B T}{\pi\hbar^2}\Gamma(s+2)\frac{\partial}{\partial\eta_F}\mathcal{F}_{s+1}(\eta_F) \qquad \text{(vii)}$$

Now work on the denominator:

$$\text{denom} = \sum_{\vec{k}} f_0 = \frac{1}{2\pi^2}\int_0^\infty f_0 2\pi k dk$$

$$\text{denom} = \frac{1}{\pi}\int_0^\infty f_0 k dk$$

$$\text{denom} = g_V \frac{m^*}{\pi\hbar^2}\int_0^\infty f_0 dE$$

$$\text{denom} = g_V \frac{m^* k_B T}{\pi\hbar^2}\int_0^\infty \frac{d\eta}{1+e^{\eta-\eta_F}}$$

$$\text{denom} = g_V \frac{m^* k_B T}{\pi\hbar^2}\Gamma(1)\mathcal{F}_0(\eta_F) \qquad \text{(viii)}$$

Using (viii) and (vii), we find

$$\langle\langle\tau_m\rangle\rangle = \frac{\text{num}}{\text{denom}} = \frac{\tau_0\Gamma(s+2)}{\Gamma(1)}\frac{\mathcal{F}_s(\eta_F)}{\mathcal{F}_0(\eta_F)} \qquad \text{(ix)}$$

Assuming non-degenerate conditions, we find

$$\boxed{\langle\langle\tau_m\rangle\rangle = \tau_0\frac{\Gamma(s+2)}{\Gamma(1)} = \tau_0\frac{\Gamma(s+2)}{\Gamma(2)}}.$$

2) Work out the corresponding result in 1D.

Solution:

The solution proceeds much as in problem 1).

$$\langle\langle\tau_m\rangle\rangle = \frac{\sum_{\vec{k}}(E-E_C)\tau_m\left(-\frac{\partial f_0}{\partial E}\right)}{(d/2)\sum_{\vec{k}} f_0} = \frac{\frac{1}{k_B T}\frac{\partial}{\partial\eta_F}\sum_{\vec{k}}(E-E_C)\tau_m f_0}{1/2\sum_{\vec{k}} f_0} = \frac{\text{num}}{\text{denom}}$$

$$\text{num} = \frac{1}{k_B T}\frac{\partial}{\partial \eta_F}\sum_{\vec{k}}\left(E-E_C\right)\tau_m f_0$$

$$= \frac{1}{k_B T}\frac{\partial}{\partial \eta_F}\int_{-\infty}^{\infty}\left(E-E_C\right)\tau_0\left(\frac{E-E_C}{k_B T}\right)^s f_0\,dk$$

$$= 2\tau_0\frac{\partial}{\partial \eta_F}\int_{0}^{\infty}\left(\frac{E-E_C}{k_B T}\right)^{s+1} f_0\,dk$$

Assuming parabolic energy bands:

$$k = \frac{\sqrt{2m^*\left(E-E_C\right)}}{\hbar} \qquad dk = \frac{\sqrt{2m^*}}{2\hbar}\left(E-E_C\right)^{-1/2}dE$$

$$\text{num} = 2\tau_0\frac{\partial}{\partial \eta_F}\int_{E_C}^{\infty}\left(\frac{E-E_C}{k_B T}\right)^{s+1} f_0\frac{\sqrt{2m^*}}{2\hbar}\left(E-E_C\right)^{-1/2}dE$$

$$\text{num} = \tau_0\frac{\sqrt{2m^*/k_B T}}{\hbar}\frac{\partial}{\partial \eta_F}\int_{E_C}^{\infty}\left(\frac{E-E_C}{k_B T}\right)^{s+1/2} f_0\,dE$$

Change variables:

$$\text{num} = \tau_0\frac{\sqrt{2m^* k_B T}}{\hbar}\frac{\partial}{\partial \eta_F}\int_{E_C}^{\infty}\frac{\eta^{s+1/2}}{1+e^{\eta-\eta_F}}d\eta$$

$$\text{num} = \tau_0\frac{\sqrt{2m^* k_B T}}{\hbar}k_B T\frac{\partial}{\partial \eta_F}\Gamma\left(s+3/2\right)\mathcal{F}_{s+1/2}\left(\eta_F\right)$$

$$\text{num} = \tau_0\frac{\sqrt{2m^* k_B T}}{\hbar}k_B T\,\Gamma\left(s+3/2\right)\mathcal{F}_{s-1/2}\left(\eta_F\right) \tag{x}$$

$$\text{denom} = \frac{1}{2}\sum_{\vec{k}} f_0 = \frac{1}{2}\int_{-\infty}^{\infty} f_0 dk$$

$$\text{denom} = \frac{1}{2}\sum_{\vec{k}} f_0 = \int_{0}^{\infty} f_0\frac{\sqrt{2m^*}}{2\hbar}\left(E-E_C\right)^{-1/2}dE$$

$$\text{denom} = \frac{\sqrt{2m^*}}{2\hbar}\int_{0}^{\infty}{}_{0}\frac{\left(E-E_C\right)^{-1/2}}{1+e^{\left(E-E_C\right)/k_B T}}dE$$

$$\text{denom} = \frac{\sqrt{2m^*/k_B T}}{2\hbar}\int_{0}^{\infty}{}_{0}\frac{\left(\frac{E-E_C}{k_B T}\right)^{-1/2}}{1+e^{\left(E-E_C\right)/k_B T}}dE$$

$$\text{denom} = \frac{\sqrt{2m^* k_B T}}{2\hbar} \int_0^\infty \frac{\eta^{-1/2}}{1+e^{\eta-\eta_F}} d\eta$$

$$\text{denom} = \frac{\sqrt{2m^* k_B T}}{2\hbar} \Gamma(1/2) \mathcal{F}_{-1/2}(\eta_F) \qquad \text{(xi)}$$

Using (x) and (xi)

$$\langle\langle \tau_m \rangle\rangle = \frac{\text{num}}{\text{denom}} = \tau_0 \frac{\Gamma(s+3/2)}{(1/2)\Gamma(1/2)} \frac{\mathcal{F}_{s-1/2}(\eta_F)}{\mathcal{F}_{-1/2}(\eta_F)}$$

$$\langle\langle \tau_m \rangle\rangle = \frac{\text{num}}{\text{denom}} = \tau_0 \frac{\Gamma(s+3/2)}{\Gamma(3/2)} \frac{\mathcal{F}_{s-1/2}(\eta_F)}{\mathcal{F}_{-1/2}(\eta_F)} \qquad \text{(xii)}$$

For non-degenerate conditions power law scattering, and 1D,

$$\boxed{\langle\langle \tau_m \rangle\rangle = \tau_0 \frac{\Gamma(s+3/2)}{\Gamma(3/2)}}.$$

3) Work out the corresponding result in 3D.

Solution:

The solution proceeds much as in problem 1) and 2).

$$\langle\langle \tau_m \rangle\rangle = \frac{\sum_{\vec{k}} (E-E_C)\tau_m \left(-\frac{\partial f_0}{\partial E}\right)}{(d/2)\sum_{\vec{k}} f_0} = \frac{\frac{1}{k_B T}\frac{\partial}{\partial \eta_F}\sum_{\vec{k}} (E-E_C)\tau_m f_0}{3/2 \sum_{\vec{k}} f_0} = \frac{\text{num}}{\text{denom}}$$

$$\text{num} = \frac{1}{k_B T}\frac{\partial}{\partial \eta_F}\sum_{\vec{k}} (E-E_C)\tau_m f_0$$

$$= \frac{1}{k_B T}\frac{\partial}{\partial \eta_F}\int_0^\infty (E-E_C)\tau_0 \left(\frac{E-E_C}{k_B T}\right)^s f_0 4\pi k^2 \, dk$$

$$= 4\pi\tau_0 \frac{\partial}{\partial \eta_F}\int_0^\infty \left(\frac{E-E_C}{k_B T}\right)^{s+1} f_0 k^2 \, dk$$

Assuming parabolic energy bands:

$$k^2 = \frac{2m^*(E-E_C)}{\hbar^2} \qquad dk = \frac{\sqrt{2m^*}}{2\hbar}(E-E_C)^{-1/2} dE \qquad k^2 dk = \frac{(2m^*)^{3/2}(E-E_C)^{1/2}}{2\hbar^3} dE$$

$$\text{num} = 4\pi\tau_0 \frac{\partial}{\partial \eta_F} \int_{E_C}^{\infty} \left(\frac{E - E_C}{k_B T} \right)^{s+1} f_0 \frac{\left(2m^*\right)^{3/2} \left(E - E_C\right)^{1/2}}{2\hbar^3} dE$$

$$\text{num} = \frac{2\pi\tau_0 \left(2m^*\right)^{3/2}}{\hbar^3} \frac{\partial}{\partial \eta_F} \int_{E_C}^{\infty} \left(\frac{E - E_C}{k_B T} \right)^{s+1} \left(E - E_C\right)^{1/2} f_0 \, dE$$

$$\text{num} = \frac{2\pi\tau_0 \left(2m^*\right)^{3/2} \sqrt{k_B T}}{\hbar^3} \frac{\partial}{\partial \eta_F} \int_{E_C}^{\infty} \left(\frac{E - E_C}{k_B T} \right)^{s+3/2} f_0 \, dE$$

Change variables:

$$\text{num} = \frac{2\pi\tau_0 \left(2m^* k_B T\right)^{3/2}}{\hbar^3} \frac{\partial}{\partial \eta_F} \int_0^{\infty} \frac{\eta^{s+3/2} d\eta}{1 + e^{\eta - \eta_F}}$$

$$\text{num} = \frac{2\pi\tau_0 \left(2m^* k_B T\right)^{3/2}}{\hbar^3} \frac{\partial}{\partial \eta_F} \Gamma\left(s + 5/2\right) \mathcal{F}_{s+3/2}\left(\eta_F\right)$$

$$\text{num} = \frac{2\pi\tau_0 \left(2m^* k_B T\right)^{3/2}}{\hbar^3} \Gamma\left(s + 5/2\right) \mathcal{F}_{s+1/2}\left(\eta_F\right) \qquad \text{(xiii)}$$

Now the denominator:

$$\text{denom} = \frac{3}{2} \sum_{\vec{k}} f_0 = \frac{3}{2} \int_0^{\infty} f_0 4\pi k^2 dk$$

$$\text{denom} = \frac{3}{2} \int_0^{\infty} f_0 4\pi k^2 dk = 3\pi \int_0^{\infty} f_0 \frac{\left(2m^*\right)^{3/2} \left(E - E_C\right)^{1/2}}{2\hbar^3} dE$$

$$\text{denom} = \frac{3\pi \left(2m^*\right)^{3/2}}{2\hbar^3} \int_0^{\infty} f_0 \left(E - E_C\right)^{1/2} dE$$

$$\text{denom} = \frac{3\pi \left(2m^* k_B T\right)^{3/2}}{2\hbar^3} \Gamma\left(3/2\right) \mathcal{F}_{1/2}\left(\eta_F\right) \qquad \text{(xiv)}$$

Using (xiii) and (xiv)

$$\left\langle\left\langle \tau_m \right\rangle\right\rangle = \frac{\text{num}}{\text{denom}} = \tau_0 \frac{2\Gamma\left(s + 5/2\right)}{3\Gamma\left(3/2\right)} \frac{\mathcal{F}_{s+1/2}\left(\eta_F\right)}{\mathcal{F}_{1/2}\left(\eta_F\right)} = \tau_0 \frac{\Gamma\left(s + 5/2\right)}{\left(3/2\right)\Gamma\left(3/2\right)} \frac{\mathcal{F}_{s+1/2}\left(\eta_F\right)}{\mathcal{F}_{1/2}\left(\eta_F\right)}$$

$$\left\langle\left\langle \tau_m \right\rangle\right\rangle = \tau_0 \frac{\Gamma\left(s + 5/2\right)}{\Gamma\left(5/2\right)} \frac{\mathcal{F}_{s+1/2}\left(\eta_F\right)}{\mathcal{F}_{1/2}\left(\eta_F\right)} \qquad \text{(xv)}$$

For non-degenerate conditions power law scattering, and 3D,

$$\boxed{\left\langle\left\langle \tau_m \right\rangle\right\rangle = \tau_0 \frac{\Gamma\left(s + 5/2\right)}{\Gamma\left(5/2\right)}}.$$

Summary of non-degenerate results:

For power law scattering, parabolic energy bands, and non-degenerate carrier statistics. the "transport average" momentum relaxation time in 1D, 2D, and 3D are:

$$\boxed{\begin{array}{l} 1D: \langle\langle \tau_m \rangle\rangle = \tau_0 \dfrac{\Gamma(s+3/2)}{\Gamma(3/2)} \\ 2D: \langle\langle \tau_m \rangle\rangle = \tau_0 \dfrac{\Gamma(s+2)}{\Gamma(2)} \\ 3D: \langle\langle \tau_m \rangle\rangle = \tau_0 \dfrac{\Gamma(s+5/2)}{\Gamma(5/2)} \end{array}} . \tag{32}$$

Knowing these times, we get the mobility from

$$\mu = \frac{q\langle\langle \tau_m \rangle\rangle}{m^*}$$

The analogous procedure in the Landauer approach is to relate the mobility to the mean-free-path according to

$$\mu = \frac{D_n}{k_B T/q} = \frac{\upsilon_T \langle\langle \lambda \rangle\rangle}{2 k_B T/q}$$

To compute $\langle\langle \lambda \rangle\rangle$, we assume power law scattering

$$\lambda(E) = \lambda_0 \left[(E - E_C)/k_B T \right]^r$$

where the characteristic exponent for the mfp is "r" rather than "s". From the definition of the average mean-free-path

$$\langle\langle \lambda \rangle\rangle \equiv \frac{\int \lambda(E) M(E) \left(-\frac{\partial f_0}{\partial E} \right) dE}{\int M(E) \left(-\frac{\partial f_0}{\partial E} \right) dE}$$

assuming parabolic bands and nondegenerate conditions, we could express $\langle\langle \lambda \rangle\rangle$ in terms of Gamma functions.

IV. Hall Factors

The definition of the Hall factor was given in (2) as

$$r_H = \frac{\langle\langle \tau_m^2 \rangle\rangle}{\langle\langle \tau_m \rangle\rangle^2} . \tag{33}$$

We have worked out the denominator in 1D, 2D, and 3D. For the numerator, we can recognize from (4) that τ_m^2 is also in power law form. From (4), we write

$$\tau_m (E - E_C)^2 = \tau_0^2 \left[(E - E_C) / k_B T \right]^{2s} , \tag{34}$$

so we can evaluate $\langle\langle \tau_m^2 \rangle\rangle$ by using the results for $\langle\langle \tau_m \rangle\rangle$ with $s \to 2s$.

Hall Factor in 1D: (non-degenerate)

$$\langle\langle \tau_m \rangle\rangle = \tau_0 \frac{\Gamma(s+3/2)}{\Gamma(3/2)}$$

$$\langle\langle \tau_m^2 \rangle\rangle = \tau_0^2 \frac{\Gamma(2s+3/2)}{\Gamma(3/2)}$$

$$r_H = \frac{\langle\langle \tau_m^2 \rangle\rangle}{\langle\langle \tau_m \rangle\rangle^2} = \frac{\tau_0^2 \dfrac{\Gamma(2s+3/2)}{\Gamma(3/2)}}{\left[\tau_0 \dfrac{\Gamma(s+3/2)}{\Gamma(3/2)} \right]^2} = \frac{\Gamma(2s+3/2)\Gamma(3/2)}{\left[\Gamma(s+3/2) \right]^2}$$

Hall Factor in 2D: (non-degenerate)

$$\langle\langle \tau_m \rangle\rangle = \tau_0 \frac{\Gamma(s+2)}{\Gamma(2)}$$

$$\langle\langle \tau_m^2 \rangle\rangle = \tau_0^2 \frac{\Gamma(2s+2)}{\Gamma(2)}$$

$$r_H = \frac{\left\langle\left\langle \tau_m^2 \right\rangle\right\rangle}{\left\langle\left\langle \tau_m \right\rangle\right\rangle^2} = \frac{\tau_0^2 \dfrac{\Gamma(2s+2)}{\Gamma(2)}}{\left[\tau_0 \dfrac{\Gamma(s+2)}{\Gamma(2)}\right]^2} = \frac{\Gamma(2s+2)\Gamma(2)}{\left[\Gamma(s+2)\right]^2}$$

Hall Factor in 3D: (non-degenerate)

$$\left\langle\left\langle \tau_m \right\rangle\right\rangle = \tau_0 \frac{\Gamma(s+5/2)}{\Gamma(5/2)}$$

$$\left\langle\left\langle \tau_m^2 \right\rangle\right\rangle = \tau_0^2 \frac{\Gamma(2s+5/2)}{\Gamma(5/2)}$$

$$r_H = \frac{\left\langle\left\langle \tau_m^2 \right\rangle\right\rangle}{\left\langle\left\langle \tau_m \right\rangle\right\rangle^2} = \frac{\tau_0^2 \dfrac{\Gamma(2s+5/2)}{\Gamma(5/2)}}{\left[\tau_0 \dfrac{\Gamma(s+5/2)}{\Gamma(5/2)}\right]^2} = \frac{\Gamma(2s+5/2)\Gamma(5/2)}{\left[\Gamma(s+5/2)\right]^2}$$

Summary of Hall factors: (non-degenerate)

$$\boxed{\begin{array}{l} 1D:\ r_H = \dfrac{\Gamma(2s+3/2)\Gamma(3/2)}{\left[\Gamma(s+3/2)\right]^2} \\ 2D:\ r_H = \dfrac{\Gamma(2s+2)\Gamma(2)}{\left[\Gamma(s+2)\right]^2} \\ 3D:\ r_H = \dfrac{\Gamma(2s+5/2)\Gamma(5/2)}{\left[\Gamma(s+5/2)\right]^2} \end{array}} \qquad (35)$$