SOLUTIONS: ECE 656 Homework (Week 4)

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1) Repeat the electron-phonon energy-momentum conservation arguments discussed in Sec. 2.5 of *Fundamentals of Carrier Transport*, but this time assume **electrons in graphene**.

Solution:
Begin with energy and momentum conservation:

$$E' = E \pm \hbar\omega \qquad (*)$$
$$\vec{p}' = \vec{p} \pm \hbar\vec{q} \qquad (**)$$

Use the dispersion of graphene to write:

$$E = \hbar\upsilon_F k = \upsilon_F p$$
$$E^2 = \left(\upsilon_F p\right)^2$$
$$E = \frac{p^2}{\left(E/\upsilon_F^2\right)} = \frac{p^2}{m^*\left(E\right)} \qquad m^*\left(E\right) \equiv \left(E/\upsilon_F^2\right)$$

Using this definition of an effective mass for graphene, (*) becomes

$$\frac{p'^2}{m^*} = \frac{p^2}{m^*} \pm \hbar\omega \qquad (*')$$

which is similar to the parabolic band result.

Now take the dot product of (**) with itself to find:

$$\vec{p}' \bullet \vec{p}' = p'^2 = p^2 \pm 2\hbar\vec{p} \bullet \vec{q} + \hbar^2 q^2$$

now divide through by m^*

$$\frac{p'^2}{m^*} = \frac{p^2}{m^*} \pm \frac{2\hbar\vec{p} \bullet \vec{q}}{m^*} + \frac{\hbar^2 q^2}{m^*}$$

and use (*')

$$\pm\hbar\omega = \pm\frac{2\hbar\vec{p} \bullet \vec{q}}{m^*} + \frac{\hbar^2 q^2}{m^*} = \pm\frac{2\hbar pq\cos\theta}{m^*} + \frac{\hbar^2 q^2}{m^*}$$

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Solving for $\hbar^2 q^2 / m^*$ we find

$$\frac{\hbar^2 q^2}{m^*} = \mp \frac{2\hbar pq\cos\theta}{m^*} \pm \hbar\omega$$

$$\hbar\beta = 2p\left[\mp\cos\theta \pm \frac{m^*\omega}{2pq}\right],$$

which is a statement of energy and momentum conservation for electrons in graphene.

Now use the dispersion of graphene and our definition of effective mass to write

$$\frac{m^*}{p} = \frac{E/\upsilon_F^2}{E/\upsilon_F} = \frac{1}{\upsilon_F}$$

which can be used to express our relation for energy-momentum conservation as

$$\boxed{\hbar q = 2p\left[\mp\cos\theta \pm \frac{\omega}{2q\upsilon_F}\right]}$$

Note that this result is almost the same as for the parabolic band result (except for the factor of 2 downstairs). The conclusions will be similar.

2) Assume a transition rate of the form:

$$S(\vec{p} \to \vec{p}\,') = C\,\delta(E - E')\,\delta(\vec{p}\,' - \vec{p} \mp \hbar\vec{q})$$

where *C* is a constant. Answer the following questions assuming parabolic energy bands.

a) Derive an expression for $|\vec{q}| = q$, which expresses conservation of energy and momentum.
b) Using the results of a), determine the minimum and maximum magnitude of $|\vec{q}|$.

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Solution:

a)
Begin with energy and momentum conservation:

$$E' = E \qquad (*)$$

$$\vec{p}' = \vec{p} \pm \hbar\vec{q} \qquad (**)$$

Since the band are parabolic, (*) can be written as:

$$\frac{p'^2}{2m^*} = \frac{p^2}{2m^*} \pm \hbar\omega \qquad (*')$$

Now take the dot product of (**) with itself to find:

$$\vec{p}' \bullet \vec{p}' = p'^2 = p^2 \pm 2\hbar\vec{p} \bullet \vec{q} + \hbar^2 q^2$$

now divide through by $2m^*$

$$\frac{p'^2}{2m^*} = \frac{p^2}{2m^*} \pm \frac{2\hbar\vec{p} \bullet \vec{q}}{2m^*} + \frac{\hbar^2 q^2}{2m^*}$$

and use (*')

$$\frac{p'^2}{2m^*} - \frac{p^2}{2m^*} = 0 = \pm\frac{2\hbar\vec{p} \bullet \vec{q}}{2m^*} + \frac{\hbar^2 q^2}{2m^*} = \pm\frac{2\hbar pq\cos\theta}{2m^*} + \frac{\hbar^2 q^2}{2m^*}$$

Solving for $\hbar q$ we find

$$\boxed{\hbar q = \mp 2p\cos\theta},$$

which is a statement of energy and momentum conservation on for elastic scattering.

b)

Note that q is the magnitude of $\vec{q}$, so it must always be greater than or equal to zero. The largest q will occur for

Absorption: $\cos\theta = -1, \quad \theta = \pi$
Emission: $\cos\theta = 1, \quad \theta = 0$

In either case, $\hbar q_{\max} = 2p$
The smallest q occurs for $\theta = \pi/2$ for both absorption and emission.
In either case, $\hbar q_{\min} = 0$
We conclude that for elastic scattering:

$$\boxed{0 \le \hbar q \le 2p}$$

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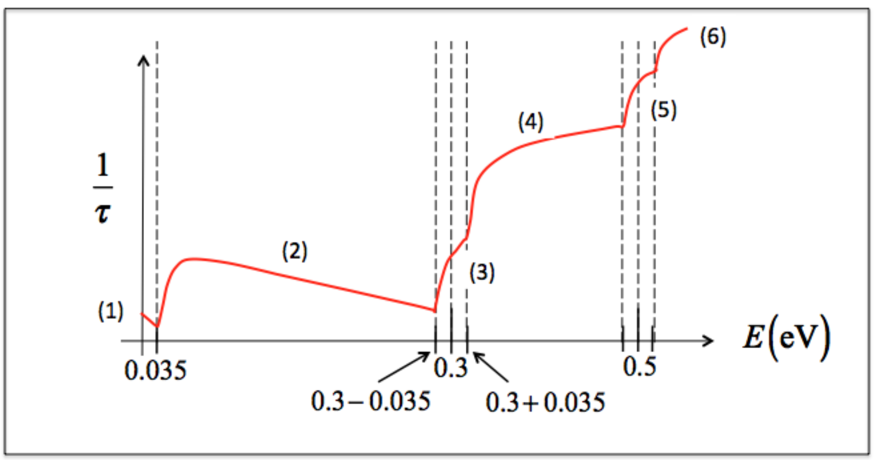
3) This problem concerns electron scattering in bulk (3D) GaAs. Assume that the optical phonon energy is $\hbar\omega_0 = 35$ meV. Recall that GaAs is a direct gap semiconductor and that the L valleys (along <111>) have energy minima that are 0.3 eV above the Γ valley minimum. The four X valleys (along <100>) have energy minima 0.5 eV above the Γ valley minimum. Recall that Γ valley electrons have a light effective mass and that the L and X valley electrons have a large (Si-like) effective mass. Answer the following questions.

Solution:

a) Sketch the total electron scattering rate vs. energy for electrons **in the Γ valley**. Label all critical energies and give a brief explanation (label absorption and emission processes separately). All energies should be referred to the bottom of the Γ valley, i..e. $E_{C\Gamma} = 0$.



The scattering rate is the sum of 6 different processes; the last five of these have thresholds for their onset.

1) POP Absorption (ABS). $\Gamma - \Gamma$ intravalley scattering.
2) POP Emission (EMS). $\Gamma - \Gamma$ intravalley scattering plus 1)
3) OP ABS. $\Gamma - \mathrm{L}$ intervalley scattering plus 1) and 2).
4) OP EMS. $\Gamma - \mathrm{L}$ intervalley scattering plus 1), 2), and 3).
5) OP ABS. $\Gamma - \mathrm{X}$ intervalley scattering plus 1), 2), 3), and 4).
6) OP EMS. $\Gamma - \mathrm{X}$ intervalley scattering plus 1), 2), 3), 4), and 5).

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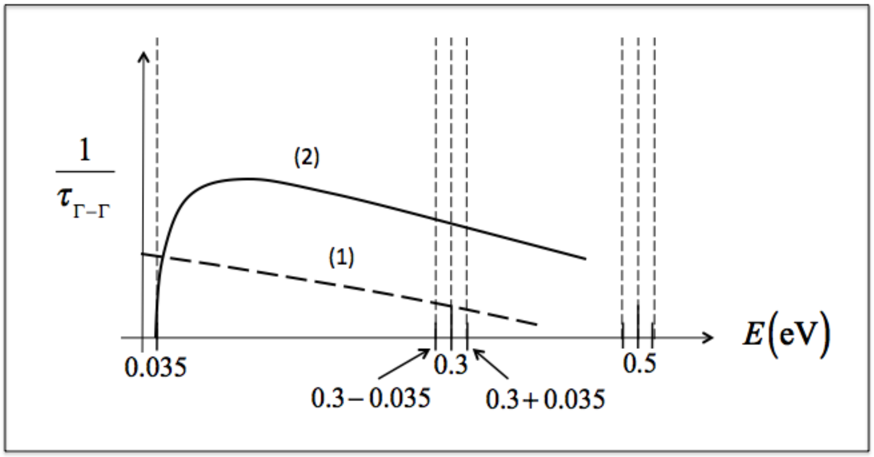
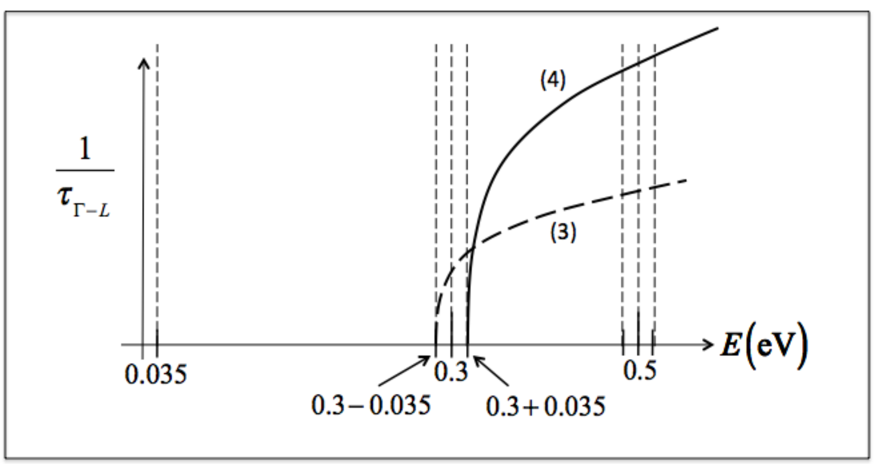
b) Sketch the Γ to Γ electron scattering rate vs. energy. Label all critical energies and give a brief explanation (label absorption and emission processes separately).



1) POP Absorption (ABS). $\Gamma-\Gamma$ intravalley scattering.
2) POP Emission (EMS). $\Gamma-\Gamma$ intravalley scattering

Note that POP ABS is proportional to N_0 and POP EMS is proportional to N_0+1, so POP EMS is larger in magnitude.

c) Sketch the Γ to L electron scattering rate vs. energy. Label all critical energies and give a brief explanation (label absorption and emission processes separately).



3) OP ABS. $\Gamma-\text{L}$ intervalley scattering.
4) OP EMS. $\Gamma-\text{L}$ intervalley scattering

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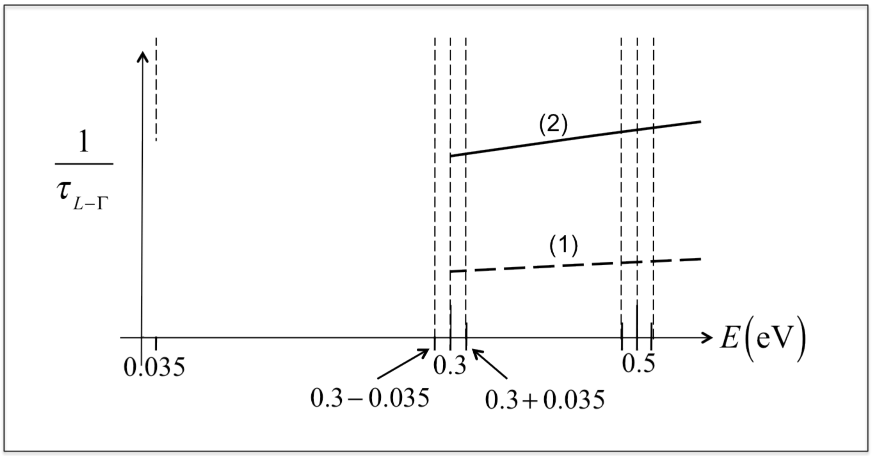
Note that EMS is, again, stronger than ABS. Also note that the scattering rates **increase** as the square root of energy because they goes as the density of final states (no POP intervalley scattering because large phonon wavevectors are needed for this type of scattering).

d) Sketch of the L to Γ electron scattering rate vs. energy. Label all critical energies and give a brief explanation (label absorption and emission processes separately).



1) OP ABS. $L-\Gamma$ intervalley scattering.
2) OP EMS. $L-\Gamma$ intervalley scattering

Note that both ABS and EMS begin at the same energy because electrons at the bottom of the L-valley can scatter down to the Γ valley by either absorbing or emitting a phonon. Also note that the magnitude of the scattering rates is smaller than in c) because the final states in this case have a valley degeneracy of 1 instead of 4 as in c). Most importantly, note that the scattering rates do not begin at zero, because they are proportional to a DOS in the final valley, where the Γ valley DOS is quite high. $1/\tau_{\Gamma-L} \propto D_{\Gamma}\left(E \pm \hbar\omega_0\right)$ begins at an energy $E = 0.3 \pm \hbar\omega_0$, where the density-of-final states (in the Γ valley) is quite high.

e) Sketch the L to L electron scattering rate vs. energy. Label all critical energies and give a brief explanation (label absorption and emission processes separately).

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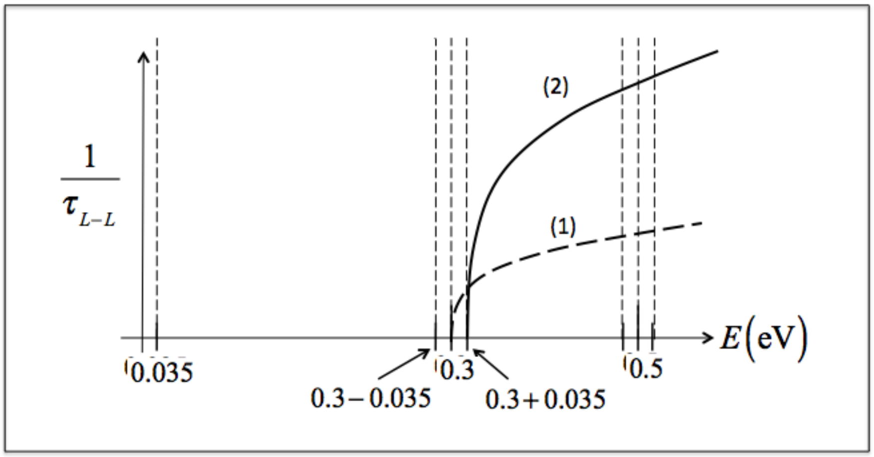


An electron at the bottom of an L-valley can absorb a phonon and scatter to another L-valley, but an electron near the bottom of an L-valley cannot scatter to another L-valley by emission, because there are no L-valley states available at that energy.

4) The deformation potential scattering rate for optical phonon emission (ODP emission) is described by:

$$\frac{1}{\tau} = \frac{2\pi}{\hbar}\left(\frac{\hbar D_0^2}{2\rho\omega_0}\right)\left(N_0+1\right)\frac{D_{3D}\left(E-\hbar\omega_0\right)}{2}.$$

Obtain the **energy flux relaxation rate** for this scattering process.

Solution:

Begin with the definition of the energy flux relaxation rate:

$$\frac{1}{\tau_{F_W}} = \sum_{\vec{p}',\uparrow} S\left(\vec{p},\vec{p}'\right)\left[\frac{F_W\left(\vec{p}\right)-F_W\left(\vec{p}'\right)}{F_W\left(\vec{p}\right)}\right] \qquad (*)$$

Work on the term in brackets and assume that the initial flux is oriented along the z-axis:

$$\left[\frac{F_W\left(\vec{p}\right)-F_W\left(\vec{p}'\right)}{F_W\left(\vec{p}\right)}\right] = \frac{E\upsilon_z-\left(E-\hbar\omega_0\right)\upsilon_z'}{E\upsilon_z} = 1-\frac{\left(E-\hbar\omega_0\right)}{E}\left(\frac{\upsilon_z'}{\upsilon_z}\right) = 1-\left(\frac{\upsilon_z'}{\upsilon_z}\right)+\frac{\hbar\omega_0}{E}\left(\frac{\upsilon_z'}{\upsilon_z}\right)$$

ECE 656 Homework (Week 4) Solutions (continued)

$$\left[\frac{F_W(\vec{p})-F_W(\vec{p}\,')}{F_W(\vec{p})}\right]=1-\left(\frac{\upsilon_z'}{\upsilon_z}\right)+\frac{\hbar\omega_0}{E}\left(\frac{\upsilon_z'}{\upsilon_z}\right)=\left(1-\frac{\upsilon_z'}{\upsilon_z}\right)-\frac{\hbar\omega_0}{E}\left(1-\frac{\upsilon_z'}{\upsilon_z}\right)+\frac{\hbar\omega_0}{E}$$

If we assume that the effective mass is constant (parabolic energy bands), then $m^*\upsilon_z = p_z$ and we find:

$$\left[\frac{F_W(\vec{p})-F_W(\vec{p}\,')}{F_W(\vec{p})}\right]=\left(1-\frac{p_z'}{p_z}\right)+\frac{\hbar\omega_0}{E}-\frac{\hbar\omega_0}{E}\left(1-\frac{p_z'}{p_z}\right)$$

Now insert this expression in (*)

$$\frac{1}{\tau_{F_W}}=\sum_{\vec{p}\,',\uparrow}S(\vec{p},\vec{p}\,')\left[\left(1-\frac{p_z'}{p_z}\right)+\frac{\hbar\omega_0}{E}-\frac{\hbar\omega_0}{E}\left(1-\frac{p_z'}{p_z}\right)\right]$$
$$=\sum_{\vec{p}\,',\uparrow}S(\vec{p},\vec{p}\,')\left(1-\frac{p_z'}{p_z}\right)+\sum_{\vec{p}\,',\uparrow}S(\vec{p},\vec{p}\,')\left(\frac{\hbar\omega_0}{E}\right)-\sum_{\vec{p}\,',\uparrow}S(\vec{p},\vec{p}\,')\frac{\hbar\omega_0}{E}\left(1-\frac{p_z'}{p_z}\right)$$

$$\frac{1}{\tau_{F_W}}=\frac{1}{\tau_m}+\left(\frac{\hbar\omega_0}{E}\right)\frac{1}{\tau}-\left(\frac{\hbar\omega_0}{E}\right)\frac{1}{\tau_m}$$

$$\boxed{\frac{1}{\tau_{F_W}}=\frac{1}{\tau_m}\left(1-\frac{\hbar\omega_0}{E}\right)+\left(\frac{\hbar\omega_0}{E}\right)\frac{1}{\tau}}$$

Since we are assuming that phonon emission occurs, this only applies for $E>\hbar\omega_0$.

5) The ODP scattering rate for 2D electrons is:

$$\left.\frac{1}{\tau_{n,n'}}\right)^{a,e}=\frac{\pi}{\hbar}\left(\frac{\hbar D_0^2}{\rho\omega_0}\right)\left(N_0+\frac{1}{2}\mp\frac{1}{2}\right)\frac{D_{2D}(E\pm\hbar\omega_0)}{2}\left(\frac{2+\delta_{n,n'}}{2}\right)$$

Let $\hbar\omega_0=1.1k_BT$, assume two subbands, and plot the absorption and emission scattering rates vs. energy for an electron in subband one.

ECE 656 Homework (Week 4) Solutions (continued)

Solution:

Let's write the scattering rate as

$$\left.\frac{1}{\tau_{n,n'}}\right)^{a,e} = \Gamma_0\left(N_0 + \frac{1}{2} \mp \frac{1}{2}\right)\left(\frac{2+\delta_{n,n'}}{2}\right)$$

where

$$\Gamma_0 = \frac{\pi}{\hbar}\left(\frac{\hbar D_0^2}{\rho\omega_0}\right)\frac{m^*}{2\pi\hbar^2}$$

$$N_0 = \frac{1}{e^{\hbar\omega_0/k_BT} - 1} = \frac{1}{e^{1.1} - 1} = \frac{1}{3-1} = \frac{1}{2}$$

Now work out the various scattering rates:

$$\left.\frac{1}{\tau_{n,n'}}\right)^{a,e} = \Gamma_0\left(\frac{1}{2} + \frac{1}{2} \mp \frac{1}{2}\right)\left(\frac{2+\delta_{n,n'}}{2}\right) = \Gamma_0\left(1 \mp \frac{1}{2}\right)\left(\frac{2+\delta_{n,n'}}{2}\right)$$

$$\left.\frac{1}{\tau_{1,1}}\right)^{a} = \Gamma_0\left(\frac{1}{2}\right)\left(\frac{3}{2}\right) = \frac{3}{4}\Gamma_0$$

$$\left.\frac{1}{\tau_{1,1}}\right)^{e} = \Gamma_0\left(\frac{3}{2}\right)\left(\frac{3}{2}\right) = \frac{9}{4}\Gamma_0$$

$$\left.\frac{1}{\tau_{1,2}}\right)^{a} = \Gamma_0\left(\frac{1}{2}\right)(1) = \frac{2}{4}\Gamma_0$$

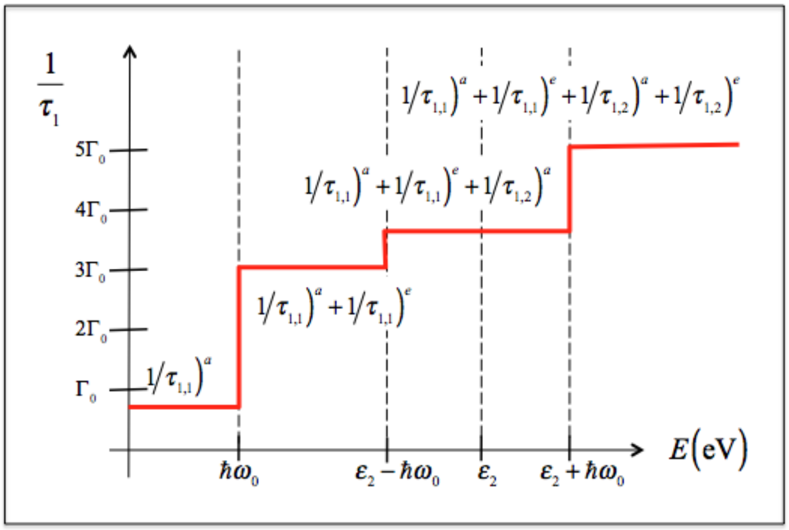
$$\left.\frac{1}{\tau_{1,2}}\right)^{e} = \Gamma_0\left(\frac{3}{2}\right)(1) = \frac{6}{4}\Gamma_0$$

Total scattering rate for $E > \varepsilon_2$

$$\left.\frac{1}{\tau_1}\right)_{tot} = \Gamma_0\left(\frac{3}{4} + \frac{9}{4} + \frac{2}{4} + \frac{6}{4}\right) = 5\Gamma_0$$

ECE 656 Homework (Week 4) Solutions (continued)

The total scattering rate for electrons in subband 1 is plotted below. In this plot, we take the zero of energy to be the bottom of the first subband, $E = \varepsilon_1 = 0$.



6) Use arguments similar to those in Sec. 2.2 of FCT and evaluate the momentum relaxation time for piezoelectric scattering.

a) Show that the scattering potential is

$$U_{PZ} = \frac{e e_{PZ}}{\kappa_S \varepsilon_0} u \qquad \text{(2.29) of FCT}$$

where e_{PZ} is the piezoelectric constant and u is the elastic wave displacement. We have replaced the charge on a electron, q, with e to avoid confusion with the phonon wave vector, $\vec{q}$.

HINT: Begin with $D = \kappa_S \varepsilon_0 \mathcal{E} + e_{PZ} \dfrac{\partial u}{\partial x}$

b) Use the scattering potential of part a) and evaluate the matrix element for PZ scattering. Show that the result is

$$\left|H_{\vec{p}',\vec{p}}\right|^2 = \left(\frac{e e_{PZ}}{\kappa_S \varepsilon_0}\right)^2 \frac{k_B T}{2 c_\ell q^2 \Omega} \delta\left(\vec{p}' - \vec{p} \mp \hbar\vec{q}\right) = \left|K_q\right|^2 \left|A_q\right|^2 \delta\left(\vec{p}' - \vec{p} \mp \hbar\vec{q}\right)$$

c) Write an expression for the transition rate, $S\left(\vec{p},\vec{p}'\right)$, and determine C_q.

d) Evaluate $1/\tau_m$ assuming that the scattering is elastic.

ECE 656 Homework (Week 4) Solutions (continued)

Solution:

a) Begin with $D = \kappa_S \varepsilon_0 \mathcal{E} + e_{PZ} \dfrac{\partial u}{\partial x}$ (*)

Assume space charge neutrality: $\nabla \bullet D = \kappa_S \varepsilon_0 \mathcal{E} + e_{PZ} \dfrac{\partial u}{\partial x} = 0$

If, $D \propto e^{iqx}$, this implies that D = 0, so we find the electric field as:

$$\mathcal{E} = -\frac{e_{PZ}}{\kappa_S \varepsilon_0} \frac{\partial u}{\partial x} = -iq \frac{e_{PZ}}{\kappa_S \varepsilon_0} u$$

The scattering potential is

$$U_S = -e \int \mathcal{E} \, dx = e \frac{e_{PZ}}{\kappa_S \varepsilon_0} u$$

$$\boxed{U_{PZ} = e \frac{e_{PZ}}{\kappa_S \varepsilon_0} u}$$

b) Evaluate the matrix element

$$H_{\vec{p}',\vec{p}} = \int \frac{e^{-i\vec{p}' \bullet \vec{r}/\hbar}}{\sqrt{\Omega}} U_S \frac{e^{i\vec{p}' \bullet \vec{r}/\hbar}}{\sqrt{\Omega}} d^3 r$$

Use the results from part a)

$$U_{PZ} = \frac{e e_{PZ}}{\kappa_S \varepsilon_0} u = K_q A_q e^{i\vec{q} \bullet \vec{r}}$$

$$K_q = \frac{e e_{PZ}}{\kappa_S \varepsilon_0} \quad \text{(eqn. 2.59d of FCT)}$$

$$H_{\vec{p}',\vec{p}} = K_q A_q \frac{1}{\Omega} \int e^{i\left(\vec{p} - \vec{p}' \pm \hbar \vec{q} \bullet \vec{r}\right)/\hbar} d^3 r$$

Following the text, FCT, we find

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$$\left|H_{\vec{p}',\vec{p}}\right|^2 = \left|K_q\right|^2 \left|A_q\right|^2 \delta\left(\vec{p}-\vec{p}'\mp\hbar\vec{q}\right) \text{ eqn. (2.60 of FCT)} \qquad (**)$$

Now quantize the lattice vibrations according to eqn. (2.71c)

$$\left|A_q\right|^2 \to \frac{\hbar}{2\rho\Omega\omega_q}\left(N_\omega + \frac{1}{2} \mp \frac{1}{2}\right)$$

and we finally write the matrix element (**) as

$$\boxed{\left|H_{\vec{p}',\vec{p}}\right|^2 = \left|K_q\right|^2 \left|A_q\right|^2 \delta\left(\vec{p}-\vec{p}'\mp\hbar\vec{q}\right) = \left(\frac{ee_{PZ}}{\kappa_S\varepsilon_0}\right)^2 \frac{\hbar}{2\rho\Omega\omega_q}\left(N_\omega + \frac{1}{2} \mp \frac{1}{2}\right)\delta\delta\left(\vec{p}-\vec{p}'\mp\hbar\vec{q}\right)} \quad (***)$$

c) Begin with the transition rate

$$S\left(\vec{p},\vec{p}'\right) = \frac{2\pi}{\hbar}\left|H_{\vec{p}',\vec{p}}\right|^2 \delta\left(\vec{p}-\vec{p}'\mp\hbar\vec{q}\right)\delta\left(E'-E\mp\hbar\omega\right)$$

Now use (***) for the matrix element and eqn. (2.66) FCT for the two delta functions to find

$$S\left(\vec{p},\vec{p}'\right) = C_q\left(N_\omega + \frac{1}{2} \mp \frac{1}{2}\right)\delta\left(E'-E\mp\hbar\omega\right)$$

where

$$C_q = \frac{2\pi}{\hbar^2}\left(\frac{ee_{PZ}}{\kappa_S\varepsilon_0}\right)^2 \frac{\hbar}{2\rho\Omega\omega}\frac{1}{\hbar\upsilon q} = \frac{2\pi}{\hbar^2}\left(\frac{ee_{PZ}}{\kappa_S\varepsilon_0}\right)^2 \frac{\hbar}{2\rho\Omega\omega}\frac{m^*}{\hbar pq}\times\frac{q}{q}$$

$$\boxed{C_q = \left(\frac{ee_{PZ}}{\kappa_S\varepsilon_0}\right)^2 \frac{\pi m^*}{\hbar\rho\upsilon_S q^2 p\Omega}} \qquad \text{(eqn. (2.73d) of FCT}$$

Here, we have used: $\omega/q = \upsilon_S$ and $p = m^*\upsilon$ for the momentum of the electron.

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d) Begin with eqn. (2.80) in FCT. Assume

$\frac{\omega}{\upsilon q} = \frac{\upsilon_S}{\upsilon} << 1$ (average electron velocity much less than the phonon velocity, so (2.80) becomes

$$\frac{1}{\tau_m} = \frac{\Omega}{4\pi^2}\int_{q_{\min}}^{q_{\max}}\left(N_\omega + \frac{1}{2} \mp \frac{1}{2}\right) C_q \frac{\hbar q}{2p}\frac{\hbar q^3}{p} dq$$

Now assume equipartition:

$$N \approx N + 1 = \frac{k_B T}{\hbar\omega}$$

$$\frac{1}{\tau_m} = \frac{\pi m^* e^2 e_{PZ}^2 k_B T}{c_\ell \kappa_S \varepsilon_0 \left(2p^3\right)}\int_{q_{\min}}^{q_{\max}} q\, dq$$

Acoustic phonon scattering is elastic: $q_{\max} = 2p$ and $q_{\min} = 0$

$$\int_{q_{\min}}^{q_{\max}} q\, dq = \frac{q_{\max}^2}{2} - \frac{q_{\min}^2}{2} = 2p^2$$

Finally, the scattering rate becomes:

$$\boxed{\frac{1}{\tau_m} = \frac{\pi m^* e^2 e_{PZ}^2 k_B T}{c_\ell \kappa_S \varepsilon_0 p} \propto \frac{1}{\sqrt{E}}}$$

The momentum relaxation rate decreases with energy – as expected for an electrostatic scattering mechanisms.

7) For alloys of compound semiconductors like $\mathrm{Al_xGa_{1\text{-}x}As}$, microscopic fluctuations in the alloy composition, x, produce perturbations in the band edges. The transition rate for allow scattering is

$$S\left(\vec{p}, \vec{p}'\right) = \frac{2\pi}{\hbar^2}\left(\frac{3\pi^2}{16}\right)\frac{\left|\Delta U\right|^2}{N\Omega}\delta\left(E' - E\right)$$

where *N* is the concentration of atoms and

ECE 656 Homework (Week 4) Solutions (continued)

$$\Delta U = x(1-x)\left(\chi_{\text{GaAs}} - \chi_{\text{AlAs}}\right)$$

with χ being the electron affinity.

a) Explain why the alloy scattering rate vanishes at $x = 0$ and at $x = 1$.

b) Derive an expression for τ_m for alloy scattering.

Solution:

a) For $x = 0$, we have GaAs, which is not an alloy, so there is no alloy scattering. For $x = 1$, we have AlAs, which us not an alloy, so there is no alloy scattering. The strongest allow scattering will occur when $x = ½$.

b)

$$S(\vec{p},\vec{p}\,') = \frac{C}{\Omega}\delta(E' - E)$$

where

$$C = \frac{2\pi}{\hbar^2}\left(\frac{3\pi^2}{16}\right)\frac{|\Delta U|^2}{N}$$

Since there is no dependence on phonon wavevector, the scattering rate and momentum relaxation rates should be equal. Since there is no dependent on β, we do not need to worry about energy-momentum conservation and can do the sum simply.

$$\frac{1}{\tau} = \frac{1}{\tau_m} = \sum_{\vec{p}\,',\uparrow} S(\vec{p},\vec{p}\,') = C\frac{1}{\Omega}\sum_{\vec{p}\,',\uparrow}\delta(E' - E)$$

We recognize the sum as one-half of the density-of-states,

$$\boxed{\frac{1}{\tau(E)} = \frac{1}{\tau_m(E)} = C\frac{D_{3D}(E)}{2}}$$

$$\tau_m(E) \propto E^{-1/2}$$

Which is power law scattering with a characteristic exponent of $s = -1/2$.

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8) Acoustic phonon scattering was assumed to be elastic when working out the momentum relaxation rate in eqn. (2.84) of FCT. Repeat the calculation but do not assume elastic scattering. Show that the result is nearly equal to eqn. (2.84) near room temperature.

Solution:

The transition rate for phonon scattering is:

$$S(\vec{p},\vec{p}\,') = \frac{2\pi}{\hbar}\left|K_q\right|^2\left|A_q\right|^2 \delta\left(\vec{p}-\vec{p}\,'\mp\hbar\vec{q}\right)\delta\left(E'-E\mp\hbar\omega\right)$$

The two delta-functions can be replaced by one that expresses energy and momentum conservation (eqn. (2.66) of FCT):

$$\delta\left(\vec{p}-\vec{p}\,'\mp\hbar\vec{q}\right)\delta\left(E'-E\mp\hbar\omega\right) \rightarrow \frac{1}{\hbar\upsilon q}\delta\left(\pm\cos\theta+\frac{\hbar q}{2p}\mp\frac{\omega_q}{\upsilon q}\right)$$

so the transition rate becomes:

$$S(\vec{p},\vec{p}\,') = \frac{2\pi}{\hbar^2\upsilon q}\left|K_q\right|^2\left|A_q\right|^2 \delta\left(\pm\cos\theta+\frac{\hbar q}{2p}\mp\frac{\omega_q}{\upsilon q}\right),$$

which is eqn. (2.67) of FCT. For ADP scattering:

$$\left|K_q\right|^2 = q^2 D_A^2 \quad \text{(eqn. (2.59a) of FCT)}$$

$$\left|A_q\right|^2 \rightarrow \frac{\hbar}{2\rho\Omega\omega_q}\left(N_{\omega_q}+\frac{1}{2}\mp\frac{1}{2}\right) \qquad \text{(eqn. (2.71c) of FCT)}$$

So we find:

$$S(\vec{p},\vec{p}\,') = \frac{\pi m^* D_A^2}{\hbar\rho p \upsilon_S}\frac{1}{\Omega}\left(N_{\omega_q}+\frac{1}{2}\mp\frac{1}{2}\right)\delta\left(\pm\cos\theta+\frac{\hbar q}{2p}\mp\frac{\omega_q}{\upsilon q}\right)$$

or

ECE 656 Homework (Week 4) Solutions (continued)

$$S(\vec{p},\vec{p}') = C_q\left(N_{\omega_q} + \frac{1}{2} \mp \frac{1}{2}\right)\delta\left(\pm\cos\theta + \frac{\hbar q}{2p} \mp \frac{\omega_q}{\upsilon q}\right) \qquad \text{(eqn. (2.72 FCT)}$$

$$C_q = \frac{\pi m^* D_A^2}{\hbar \rho p \upsilon_S}\frac{1}{\Omega} \qquad \text{(eqn. (2.73a) FCT)}$$

Following FCT, we write the momentum relaxation rate as:

$$\frac{1}{\tau_m} = \sum_{\vec{q},\uparrow} S(\vec{p},\vec{p}')\frac{(\mp\hbar q\cos\theta)}{p} = \frac{\Omega}{8\pi^3}\int_o^{2\pi} d\phi \int_0^{\infty}\int_{-1}^{+1} S(\vec{p},\vec{p}')\frac{(\mp\hbar q\cos\theta)}{p} d(\cos\theta) q^2 dq$$

or

$$\frac{1}{\tau_m} = \frac{\Omega}{4\pi^2}\int_0^{\infty}\int_{-1}^{+1} C_q\left(N_{\omega_q} + \frac{1}{2} \mp \frac{1}{2}\right)\delta\left(\pm\cos\theta + \frac{\hbar q}{2p} \mp \frac{\omega_q}{\upsilon q}\right)\frac{(\mp\hbar q\cos\theta)}{p} d(\cos\theta) q^2 dq$$

Now assume equipartition: $N_{\omega_q} \approx N_{\omega_q} + 1 \approx \frac{k_B T}{\hbar\omega_q}$ and use the expression for C_β.

$$\frac{1}{\tau_m} = \frac{m^* D_A^2 k_B T}{4\pi\hbar\rho p^2 \upsilon_S^2}\int_0^{\infty}\int_{-1}^{+1}\delta\left(\pm\cos\theta + \frac{\hbar q}{2p} \mp \frac{\omega_q}{\upsilon q}\right)(\mp\cos\theta) d(\cos\theta) q^2 dq$$

Integrate over β first:

$$\frac{1}{\tau_m} = \frac{m^* D_A^2 k_B T}{4\pi\hbar\rho p^2 \upsilon_S^2}\int_{q_{\min}}^{q_{\max}}\left(\frac{\hbar q}{2p} \mp \frac{\omega_q}{\upsilon q}\right) q^2 dq = \frac{m^* D_A^2 k_B T}{4\pi\hbar\rho p^2 \upsilon_S^2}\int_{q_{\min}}^{q_{\max}}\left(\frac{\hbar q}{2p} \mp \frac{\upsilon_S}{\upsilon}\right) q^2 dq$$

where we have used the acoustic phonon dispersion for the last step.

$$\frac{1}{\tau_m} = \frac{m^* D_A^2 k_B T}{4\pi\hbar\rho p^2 \upsilon_S^2}\left[\frac{\hbar}{2p}\left(\frac{q_{\max}^4}{4} - \frac{q_{\min}^4}{4}\right) \mp \frac{\upsilon_S}{\upsilon}\left(\frac{q_{\max}^3}{3} - \frac{q_{\min}^3}{3}\right)\right]$$

ECE 656 Homework (Week 4) Solutions (continued)

Assume $q_{\max} >> q_{\min}$ and $q_{\max}^4 >> q_{\max}^3$

$$\frac{1}{\tau_m} = \frac{m^* D_A^2 k_B T}{8\pi\rho p^3 \upsilon_S^2}\left(\frac{q_{\max}^4}{4}\right)$$

Energy and momentum conservation gives (see FCT, eqn. 2.54)

$$\hbar q_{\max} = 2p\left(1 \pm \frac{\upsilon_S}{\upsilon}\right)$$

$$\frac{q_{\max}^4}{4} = \frac{4p^4}{\hbar^4}\left(1 \pm \frac{\upsilon_S}{\upsilon}\right)^4$$

so the momentum relaxation rate becomes:

$$\frac{1}{\tau_m} = \frac{m^* p D_A^2 k_B T}{2\pi\hbar^4 \rho \upsilon_S^2}\left(1 \pm \frac{\upsilon_S}{\upsilon}\right)^4 = \frac{m^* p D_A^2 k_B T}{2\pi\hbar^4 c_\ell}\left(1 \pm \frac{\upsilon_S}{\upsilon}\right)^4$$

Recall that the 3D density-of-states is:

$$D_{3D}(E) = \frac{\left(2m^*\right)^{3/2}}{4\pi^2\hbar^3} E^{1/2}$$

which can be used to re-express the momentum relaxation rate as:

$$\boxed{\frac{1}{\tau_m(E)} = \frac{m^* D_A^2 k_B T}{\hbar c_\ell}\frac{D_{3D}(E)}{2}\left(1 \pm \frac{\upsilon_S}{\upsilon}\right)^4},$$

which is almost exactly the result for elastic scattering (eqn. (2.84) of FCT) since for a typical electron $\upsilon << \upsilon_S$.