SOLUTIONS: ECE 656 Homework (Week 6)

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1) We have discussed $M(E)$ for a 3D semiconductor with parabolic energy bands. Answer the following two questions about a 3D semiconductor with non-parabolic energy bands.

   a) Assume that the non-parabolicity can be described by

   $$E(1+\alpha E)=\frac{\hbar^2 k^2}{2m^*(0)}.$$

   Derive an expression for the corresponding $M(E)$.

   b) Using the following numbers for GaAs,

   $m^*(0)=0.067\,m_0$
   $\alpha=0.64$,

   plot $M(E)$ from the bottom of the Γ valley to $E=0.3$ eV comparing results from the non-parabolic expression derived in part a) to the parabolic expression.

Solutions:

We know that non-parabolicity flattens the bands – it increases the DOS and lowers the velocity, so it is not obvious whether $M(E)$ will be larger or smaller, since $M(E)$ is the product of the DOS and velocity. Let's work it out and see what the answer is.

1a)

Begin with the definition:

$$M_{3D}(E)=\frac{h}{4}\langle \upsilon_x^+\rangle D_{3D}(E) \qquad (1)$$

$$\langle \upsilon_x^+\rangle=(1/2)\upsilon(E) \qquad (2)$$

Step 1: compute $D_{3D}(E)$ for non-parabolic bands
Step 2: compute $\upsilon(E)$ for non-parabolic bands
Step 3: multiply the two to get the answer

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Step 1:

$$D_{3D}(E)dE = \frac{N_{3D}(k)}{\Omega} 4\pi k^2 dk = \frac{1}{4\pi^3} 4\pi k^2 dk = \frac{1}{\pi^2} k^2 dk \tag{3}$$

$$E(1+\alpha E) = \frac{\hbar^2 k^2}{2m^*(0)} \tag{4}$$

Solve for *k*:

$$k = \frac{\sqrt{2m^*(0)E(1+\alpha E)}}{\hbar} \tag{5}$$

differentiate to find:

$$kdk = \frac{m^*(0)}{\hbar^2}(1+2\alpha E)dE \tag{6}$$

put the two together:

$$k^2 dk = \frac{m^*(0)}{\hbar^3}\sqrt{2m^*(0)E(1+\alpha E)}(1+2\alpha E)dE$$

Insert in (3)

$$D_{3D}(E)dE = \frac{1}{\pi^2}k^2 dk = \frac{m^*(0)}{\pi^2\hbar^3}\sqrt{2m^*(0)E(1+\alpha E)}(1+2\alpha E)dE$$

$$D_{3D}(E) = \frac{1}{\pi^2}k^2 dk = \frac{m^*(0)}{\pi^2\hbar^3}\sqrt{2m^*(0)E(1+\alpha E)}(1+2\alpha E) \tag{7}$$

Step 2:

From (6)

$$\frac{dE}{dk} = \frac{\hbar^2 k}{m^*(0)}\frac{1}{(1+2\alpha E)} \rightarrow \frac{1}{\hbar}\frac{dE}{dk} = \upsilon = \frac{\hbar k}{m^*(0)}\frac{1}{(1+2\alpha E)}$$

Now use (5)

$$\upsilon(E) = \frac{\hbar k}{m^*(0)}\frac{1}{(1+2\alpha E)} = \sqrt{\frac{2E(1+\alpha E)}{m^*(0)}}\frac{1}{(1+2\alpha E)} \tag{8}$$

ECE 656 Homework (Week 6) Solutions (continued)

Step 3:

Now use (1), (2), (7), and (8)

$$M_{3D}(E)=\frac{h}{4}\left(\frac{\upsilon(E)}{2}\right)D_{3D}(E)=\frac{h}{4}\left(\frac{\upsilon(E)}{2}\right)D_{3D}(E)=\frac{h}{4}\sqrt{\frac{E(1+\alpha E)}{2m^*(0)}}\frac{1}{(1+2\alpha E)}D_{3D}(E)$$

$$M_{3D}(E)=\frac{h}{4}\sqrt{\frac{E(1+\alpha E)}{2m^*(0)}}\frac{1}{(1+2\alpha E)}\left\{\frac{m^*(0)}{\pi^2\hbar^3}\sqrt{2m^*(0)E(1+\alpha E)}(1+2\alpha E)\right\}$$

$$M_{3D}(E)=\frac{m^*(0)}{2\pi\hbar^2}E(1+\alpha E)$$

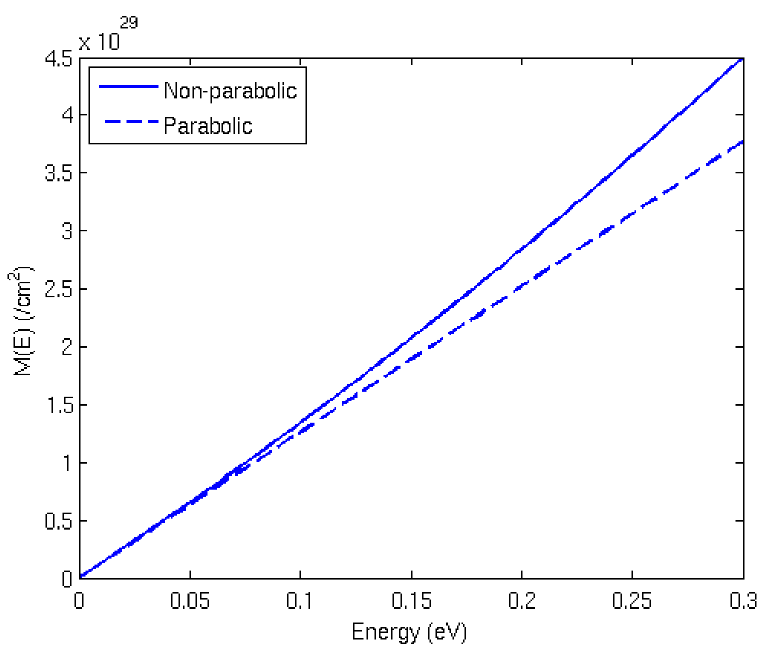
We have been assuming that the bottom of the conduction band is at $E_C=0$. Let's put E_C back in explicitly.

$$\boxed{M_{3D}(E)=\frac{m^*(0)}{2\pi\hbar^2}(E-E_C)\left[1+\alpha(E-E_C)\right]}$$

So we see that the $M(E)$ increases when the bands are nonparabolic.

1b) The following Matlab script is used to produce the plot below.

```
set(get(gca,'ylabel'),'FontSize',font_size);
```



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```
function ece656_hw6_1b()

% ECE 656 (Fall 2013) HW#6, 1b
% This script plots and compares the density of modes for GaAs assuming parabolic
% non-parabolic energy bands
```

Parameters

```
m_star = 0.067; % (1) effective mass coefficient
alpha = 0.64; % (/eV) non-parabolicty factor

m_o = 0.511e6; % (eV/c^2) electron rest mass
hbar = 6.582e-16; % (eV-s)
m = m_star*m_o; % (eV/c^2) electron effective mass

E = linspace(0,0.3,100); % (eV) range of energy
```

Calculation of Modes

```
M_3D_nonp = m./(2*pi*hbar^2).*E.*(1+alpha.*E).*1e-4; % (/cm^2)
M_3D_p = m./(2*pi*hbar^2).*E.*1e-4; % (/cm^2)
```

Plots

```
figure(1)

plot(E, M_3D_nonp,'LineWidth',2)
hold on
plot(E, M_3D_p,'--','LineWidth',2)

xlabel('Energy (eV)');
ylabel('M(E) (/cm^2)');
legend('Non-parabolic', 'Parabolic','Location','NorthWest');

xlim([0 max(E)])

font_size = 12;

set(gcf, 'color', 'white');
set(gca, 'FontSize', font_size);
set(get(gca,'title'), 'FontSize',font_size);
set(get(gca,'xlabel'),'FontSize',font_size);
```

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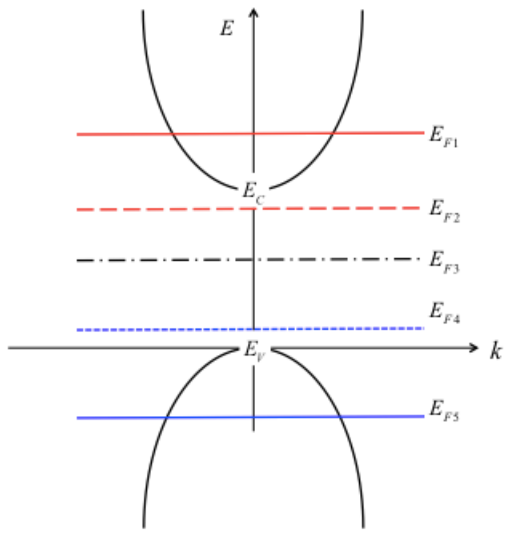
2) The figure below shows a semiconductor with the Fermi level located in five different locations. If we use the Landauer expression to compute the current:

$$I = (2q/h)\int_{E_1}^{E_2} \mathcal{T}(E) M(E)(f_1 - f_2)\, dE$$

what are appropriate limits of integration, E_1 and E_2, for each case? You may assume room temperature, a bandgap of 1 eV, and that $E_{F1} \approx E_{F2} \approx E_F$.



Solution:

Case 1: Only states in the conduction band are involved, so

$$\boxed{I = (2q/h)\int_{E_C}^{\infty} \mathcal{T}(E) M(E)(f_1 - f_2)\, dE}$$

We could safely replace ∞ with $E_F + 5k_BT$.

Case 2: Again, only states in the conduction band are involved, so

$$\boxed{I = (2q/h)\int_{E_C}^{\infty} \mathcal{T}(E) M(E)(f_1 - f_2)\, dE}$$

We could safely replace ∞ with $E_C + 5k_BT$.

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Case 3: There are not many electrons or holes involved, but we need to include both bands.

$$\boxed{I = (2q/h)\int_{-\infty}^{+\infty} \mathcal{T}(E)M(E)(f_1 - f_2)dE}$$

We could safely replace $+\infty$ with $E_C + 5k_BT$ and $-\infty$ with $E_V - 5k_BT$.

Case 4: Only states in the valence band are involved, so

$$\boxed{I = (2q/h)\int_{-\infty}^{E_V} \mathcal{T}(E)M(E)(f_1 - f_2)dE}$$

We could safely replace $-\infty$ with $E_V - 5k_BT$.

Case 5: Again, only states in the valence band are involved, so

$$\boxed{I = (2q/h)\int_{-\infty}^{E_V} \mathcal{T}(E)M(E)(f_1 - f_2)dE}$$

We could safely replace $-\infty$ with $E_F - 5k_BT$.

3) Determine the limits of integration, E_1 and E_2, for the integral in the Landauer expression:

$$I = (2q/h)\int_{E_1}^{E_2} \mathcal{T}(E)M(E)(f_1 - f_2)dE$$

for the case of T = 0 K. Assume that contact one is grounded and that a positive voltage (not necessarily small) has been applied to contact 2.

Solution:

In the left contact, states below E_{F1} are occupied, and in the right contact, states below $E_{F2} = E_{F1} - qV$ are occupied. The only ranges where $(f_1 - f_2)$ is $E_{F2} < E < E_{F1}$, so

$$\boxed{I = (2q/h)\int_{E_{F1-qV}}^{E_{F1}} \mathcal{T}(E)M(E)(f_1 - f_2)dE}$$

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4) The ballistic conductance is often derived from a k-space treatment, which writes the current from left to right as

$$I^{+}=\frac{1}{L}\sum_{k>0} q\upsilon_x f_0\left(E_{F1}\right)$$

and the current from right to left as

$$I^{-}=\frac{1}{L}\sum_{k<0} q\upsilon_x f_0\left(E_{F2}\right)$$

The net current is the difference between the two. In the ballistic limit, the Landauer expression for the current is

$$I=\left(2q/h\right)\int_{E_1}^{E_2} M\left(E\right)\left(f_1-f_2\right)dE$$

4a) Assume parabolic energy bands, evaluate the net current from the k-space approach, and show that it is the same as the Landauer expression.

Solution:

$$I^{+}=\frac{1}{L}\sum_{k>0} q\upsilon_x f_0\left(E_{F1}\right)=\frac{1}{L}\frac{L}{2\pi}\times 2\int_0^{+\infty} q\, f_0\left(E_{F1}\right)dk$$

$$I^{+}=\frac{q}{\pi}\times\int_0^{+\infty} f_0\left(E_{F1}\right)dk=\frac{q}{\pi}\int_0^{+\infty} f_0\left(E_{F1}\right)\frac{dk}{dE}dE$$

$$E=\frac{\hbar^2k^2}{2m^*}\qquad dE=\frac{\hbar^2 kdk}{m^*}\qquad dk=\frac{m^*dE}{\hbar^2 k}\qquad k=\frac{\sqrt{2m^*E}}{\hbar}\qquad \frac{dk}{dE}=\frac{m^*}{\hbar\sqrt{2m^*E}}=\frac{1}{\hbar}\sqrt{\frac{m^*}{2E}}$$

$$I^{+}=q\int_0^{+\infty}\upsilon\left(\frac{1}{\pi\hbar}\sqrt{\frac{m^*}{2E}}\right)f_0\left(E_{F1}\right)dE$$

$$I^{+}=\frac{2q}{h}\int_0^{+\infty}\frac{h}{4}\upsilon\left(\frac{2}{\pi\hbar}\sqrt{\frac{m^*}{2E}}\right)f_0\left(E_{F1}\right)dE=\frac{2q}{h}\int_0^{+\infty}\left\{\frac{h}{4}\upsilon D_{1D}\left(E\right)\right\}f_0\left(E_{F1}\right)dE$$

$$I^{+}=\frac{2q}{h}\int_0^{+\infty} M\left(E\right)f_0\left(E_{F1}\right)dE$$

where

$$M\left(E\right)\equiv\frac{h}{4}\left\langle\upsilon_x^{+}\right\rangle D_{1D}\left(E\right)\qquad\qquad \left\langle\upsilon_x^{+}\right\rangle=\upsilon$$

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Similarly, we can show:

$$I^- = \frac{2q}{h}\int_0^{+\infty} M(E) f_0(E_{F2}) dE$$

The net current is

$$I = I^+ - I^- = \frac{2q}{h}\int_0^{+\infty} M(E)\left[f_0(E_{F1}) - f_0(E_{F2}) \right] dE$$

$$\boxed{I = \frac{2q}{h}\int_0^{+\infty} M(E)(f_1 - f_2) dE}$$

Note that in solving this problem, we have also defined $M(E)$ and $\left\langle \upsilon_x^+ \right\rangle$ in 1D with parabolic energy bands.

4b) Assume parabolic energy bands, but now assume **2D electrons**. Evaluate the net current from the k-space approach, and show that it is the same as the Landauer expression.

Solution:

$$I^+ = \frac{1}{A}\sum_{k_x>0} q\upsilon_x f_0(E_{F1}) = \frac{1}{A}\frac{A}{(2\pi)^2}\times 2\int_0^{+\infty}\int_{-\pi/2}^{+\pi/2} q\upsilon\cos\theta\, f_0(E_{F1}) k\, dk\, d\theta$$

$$\int_{-\pi/2}^{+\pi/2}\cos\theta\, d\theta = 2$$

$$I^+ = \frac{q}{2\pi^2}\int_0^{+\infty} 2\upsilon\, f_0(E_{F1}) k\, dk = \frac{q}{2\pi^2}\int_0^{+\infty} 2\upsilon\, f_0(E_{F1})\frac{kdk}{dE} dE$$

$$E = \frac{\hbar^2 k^2}{2m^*} \qquad dE = \frac{\hbar^2 kdk}{m^*} \qquad kdk = \frac{m^* dE}{\hbar^2} \qquad \frac{kdk}{dE} = \frac{m^*}{\hbar^2}$$

$$I^+ = \frac{q}{2}\int_0^{+\infty}\frac{2\upsilon}{\pi} f_0(E_{F1})\left(\frac{m^*}{\pi\hbar^2}\right) dE = \frac{q}{2}\int_0^{+\infty}\left\langle \upsilon_x^+ \right\rangle f_0(E_{F1}) D_{2D} dE$$

where

$$\left\langle \upsilon_x^+ \right\rangle = \frac{2}{\pi}\upsilon$$

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$$I^+ = \frac{q}{2}\int_0^{+\infty} \left\langle \upsilon_x^+ \right\rangle D_{2D} f_0\left(E_{F1}\right) dE = I^+ = \frac{2q}{h}\int_0^{+\infty} \left\{ \frac{h}{4}\left\langle \upsilon_x^+ \right\rangle D_{2D} \right\} f_0\left(E_{F1}\right) dE$$

$$I^+ = \frac{2q}{h}\int_0^{+\infty} M(E) f_0(E_{F1}) dE$$

where

$$M(E) \equiv \frac{h}{4}\left\langle \upsilon_x^+ \right\rangle D_{2D}(E)$$

Similarly, we can show:

$$I^- = \frac{2q}{h}\int_0^{+\infty} M(E) f_0\left(E_{F2}\right) dE$$

The net current is

$$I = I^+ - I^- = \frac{2q}{h}\int_0^{+\infty} M(E)\left[f_0\left(E_{F1}\right) - f_0\left(E_{F2}\right) \right] dE$$

$$\boxed{I = \frac{2q}{h}\int_0^{+\infty} M(E)\left(f_1 - f_2\right) dE}$$

Note that in solving this problem, we have also defined $M(E)$ and $\left\langle \upsilon_x^+ \right\rangle$ in 2D with parabolic energy bands.

4c) Assume parabolic energy bands, but now assume 3D electrons. Evaluate the net current from the k-space approach, and show that it is the same as the Landauer expression.

Solution:

$$I^+ = \frac{1}{\Omega}\sum_{k_x>0} q\upsilon_x f_0\left(E_{F1}\right) = \frac{1}{\Omega}\frac{\Omega}{(2\pi)^3} \times 2\int_0^{+\infty}\int_0^{\pi}\int_{-\pi/2}^{+\pi/2} q\upsilon \sin\theta \cos\phi\, f_0\left(E_{F1}\right) k^2 \sin\theta\, d\theta\, d\phi\, dk$$

$$I^+ = \frac{q}{4\pi^3}\int_0^{+\infty}\int_0^{\pi}\int_{-\pi/2}^{+\pi/2} \upsilon \sin\theta \cos\phi\, f_0\left(E_{F1}\right) k^2 \sin\theta\, d\theta\, d\phi\, dk$$

$$\int_{-\pi/2}^{+\pi/2} \cos\phi\, d\phi = 2$$

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$$I^{+}=\frac{q}{2\pi^{3}}\int_{0}^{+\infty}\int_{0}^{\pi}\upsilon\sin\theta\, f_{0}\left(E_{F1}\right)k^{2}\sin\theta\, d\theta\, dk$$

$$\int_{0}^{\pi}\sin^{2}\theta\, d\theta=\frac{\pi}{2}$$

$$I^{+}=\frac{q}{4\pi^{2}}\int_{0}^{+\infty}\upsilon\, f_{0}\left(E_{F1}\right)k^{2}\, dk=\frac{q}{4\pi^{2}}\int_{0}^{+\infty}\upsilon\, f_{0}\left(E_{F1}\right)k^{2}\,\frac{dk}{dE}dE$$

$$E=\frac{\hbar^{2}k^{2}}{2m^{*}}\qquad dE=\frac{\hbar^{2}kdk}{m^{*}}\qquad kdk=\frac{m^{*}dE}{\hbar^{2}}\qquad k=\frac{\sqrt{2m^{*}E}}{\hbar}$$

$$k^{2}dk=\frac{\sqrt{2m^{*}E}}{\hbar}\frac{m^{*}dE}{\hbar^{2}}=\frac{m^{*}\sqrt{2m^{*}E}}{\hbar^{3}}dE$$

$$I^{+}=\frac{q}{4\pi^{2}}\int_{0}^{+\infty}\upsilon\, f_{0}\left(E_{F1}\right)k^{2}\,\frac{dk}{dE}dE=\frac{q}{4\pi^{2}}\int_{0}^{+\infty}\upsilon\, f_{0}\left(E_{F1}\right)\frac{m^{*}\sqrt{2m^{*}E}}{\hbar^{3}}dE$$

$$I^{+}=\frac{q}{4}\int_{0}^{+\infty}\upsilon\left\{\frac{m^{*}\sqrt{2m^{*}E}}{\pi^{2}\hbar^{3}}\right\}f_{0}=\frac{q}{4\pi}\int_{0}^{+\infty}\upsilon\, D_{3D}\left(E\right)f_{0}\left(E_{F1}\right)dE$$

$$I^{+}=\frac{q}{2}\int_{0}^{+\infty}\left(\frac{\upsilon}{2}\right)D_{3D}\left(E\right)f_{0}\left(E_{F1}\right)dE=\frac{q}{2\pi}\int_{0}^{+\infty}\left\langle\upsilon_{x}^{+}\right\rangle D_{3D}\left(E\right)f_{0}\left(E_{F1}\right)dE$$

where

$$\left\langle\upsilon_{x}^{+}\right\rangle=\frac{\upsilon}{2}$$

$$I^{+}=\frac{2q}{h}\int_{0}^{+\infty}\left\{\frac{h}{4}\left\langle\upsilon_{x}^{+}\right\rangle D_{3D}\left(E\right)\right\}f_{0}\left(E_{F1}\right)dE$$

$$I^{+}=\frac{2q}{h}\int_{0}^{+\infty}M\left(E\right)f_{0}(E_{F1})dE$$

where

$$M\left(E\right)\equiv\frac{h}{4}\left\langle\upsilon_{x}^{+}\right\rangle D_{3D}\left(E\right)$$

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Similarly, we can show:

$$I^- = \frac{2q}{h}\int_0^{+\infty} M(E) f_0(E_{F2})dE$$

The net current is

$$I = I^+ - I^- = \frac{2q}{h}\int_0^{+\infty} M(E)\left[f_0(E_{F1}) - f_0(E_{F2})\right]dE$$

$$\boxed{I = \frac{2q}{h}\int_0^{+\infty} M(E)(f_1 - f_2)dE}$$

Note that in solving this problem, we have also defined $M(E)$ and $\langle \upsilon_x^+ \rangle$ in 3D with parabolic energy bands.

5) The quantity,

$$\langle M \rangle = \int_{E_C}^{\infty} M(E)\left(-\frac{\partial f_0}{\partial E}\right)dE$$

is the number of conduction band channels in the Fermi-Window. Answer the following questions.

5a) Evaluate $\langle M \rangle$ for an arbitrary temperature and location of the Fermi level assuming a 2D semiconductor with parabolic energy bands.

Solution:

$$\langle M \rangle = \int_{E_C}^{\infty} M(E)\left(-\frac{\partial f_0}{\partial E}\right)dE = \frac{\partial}{\partial E_F}\int_{E_C}^{\infty} M(E) f_0(E)dE$$

$$M(E) = W\, M_{2D}(E) = W\frac{\sqrt{2m^*(E-E_C)}}{\pi\hbar}$$

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$$\langle M \rangle = \frac{\partial}{\partial E_F} \int_{E_C}^{\infty} W \frac{\sqrt{2m^*(E-E_C)}}{\pi\hbar} \frac{1}{1+e^{(E-E_F)/k_BT}} dE$$

$$\langle M \rangle = W \frac{\sqrt{2m^*}}{\pi\hbar} \frac{\partial}{\partial E_F} \int_{E_C}^{\infty} \frac{\sqrt{(E-E_C)}}{1+e^{(E-E_C+E_C-E_F)/k_BT}} dE$$

$$\langle M \rangle = W \frac{\sqrt{2m^*}}{\pi\hbar} (k_BT)^{3/2} \frac{\partial}{\partial E_F} \int_{0}^{\infty} \frac{\sqrt{\eta}}{1+e^{\eta-\eta_F}} d\eta$$

where

$\eta = (E-E_C)/k_BT$ and $\eta_F = (E_F-E_C)/k_BT$

$$\langle M \rangle = W \frac{\sqrt{2m^*}}{\pi\hbar} (k_BT)^{1/2} \frac{\partial}{\partial \eta_F} \left\{ \frac{\sqrt{\pi}}{2} \mathcal{F}_{1/2}(\eta_F) \right\} = W \sqrt{\frac{m^*}{2\pi}} \frac{1}{\hbar} (k_BT)^{1/2} \mathcal{F}_{-1/2}(\eta_F)$$

$$\boxed{\langle M \rangle = W \sqrt{\frac{m^*}{2\pi}} \frac{1}{\hbar} (k_BT)^{1/2} \mathcal{F}_{-1/2}(\eta_F)}$$

5b) Evaluate $\langle M \rangle$ for an arbitrary temperature and location of the Fermi level above the Dirac point, E_D, assuming graphene.

Solution:

The DOS is:

$$D_{2D}(E) = \frac{2(E-E_D)}{\pi\hbar^2\upsilon_F^2}$$

$$M(E) = W\frac{h}{4}\langle \upsilon_x^+ \rangle D_{2D}(E) = W\frac{h}{4}\left(\frac{2}{\pi}\upsilon_F\right)\frac{2(E-E_D)}{\pi\hbar^2\upsilon_F^2} = W\frac{4}{\upsilon_F h}(E-E_D)$$

$$\langle M \rangle = \int_{E_D}^{\infty} M(E)\left(-\frac{\partial f_0}{\partial E}\right)dE = \frac{\partial}{\partial E_F}\int_{E_D}^{\infty} M(E) f_0(E) dE$$

$$\langle M \rangle = W\frac{4}{\upsilon_F h}\frac{\partial}{\partial E_F}\int_{E_D}^{\infty}(E-E_D)\frac{1}{1+e^{(E-E_F)/k_BT}}dE$$

$$\langle M \rangle = W\frac{4}{\upsilon_F h}(k_BT)^2\frac{\partial}{\partial E_F}\int_{0}^{\infty}\frac{\eta}{1+e^{\eta-\eta_F}}d\eta$$

$$\langle M \rangle = W\frac{4}{\upsilon_F h}(k_BT)\frac{\partial}{\partial \eta_F}\mathcal{F}_1(\eta_F)$$

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$$\boxed{\langle M\rangle = W\frac{4}{\upsilon_F h}(k_BT)\mathcal{F}_0(\eta_F)}$$

5c) Assume that $E_F = E_C + 0.1\,\text{eV} = E_D + 0.1\,\text{eV}$. For 5a), assume Si with $m^* = 0.19m_0$ and a valley degeneracy of 2. For 5b), assume graphene parameters, $\upsilon_F = 10^8$ cm/s and a valley degeneracy of 2. Compare the numerical values of $\langle M\rangle$ for these two cases assuming T = 300 K.

Solution:

$$\frac{\langle M\rangle_{\text{Si}}}{\langle M\rangle_{\text{graphene}}} = \frac{g_V W\sqrt{\dfrac{m^*}{2\pi}}\dfrac{1}{\hbar}(k_BT)^{1/2}\mathcal{F}_{-1/2}(\eta_F)}{g_v W\dfrac{4}{\upsilon_F h}(k_BT)\mathcal{F}_0(\eta_F)}$$

$$\frac{\langle M\rangle_{\text{Si}}}{\langle M\rangle_{\text{graphene}}} = \frac{\upsilon_F}{2}\sqrt{\frac{\pi m^*}{2k_BT}}\left(\frac{\mathcal{F}_{-1/2}(\eta_F)}{\mathcal{F}_0(\eta_F)}\right)$$

$$\frac{\langle M\rangle_{\text{Si}}}{\langle M\rangle_{\text{graphene}}} = \frac{\upsilon_F}{2\upsilon_T}\left(\frac{\mathcal{F}_{-1/2}(\eta_F)}{\mathcal{F}_0(\eta_F)}\right)$$

where

$$\upsilon_T = \sqrt{\frac{2k_BT}{\pi m^*}} = 1.23\times 10^7 \text{ cm/s}$$

$$\frac{\langle M\rangle_{\text{Si}}}{\langle M\rangle_{\text{graphene}}} = \frac{10^8}{2.47\times 10^7}\left(\frac{\mathcal{F}_{-1/2}(\eta_F)}{\mathcal{F}_0(\eta_F)}\right) = 4.05\times\left(\frac{\mathcal{F}_{-1/2}(\eta_F)}{\mathcal{F}_0(\eta_F)}\right)$$

$$\eta_F = 0.1/0.026 = 3.85$$

$$\frac{\langle M\rangle_{\text{Si}}}{\langle M\rangle_{\text{graphene}}} = 4.05\times\left(\frac{\mathcal{F}_{-1/2}(3.85)}{\mathcal{F}_0(3.85)}\right) = 4.05\times\left(\frac{2.1392}{3.821}\right) = 2.27$$

$$\boxed{\frac{\langle M\rangle_{\text{Si}}}{\langle M\rangle_{\text{graphene}}} = 2.27}$$

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6) For a 3D diffusive resistor, we relate the current density to the electric field by

$\mathcal{E}_x = \rho_{3D} J_x$ V/m,
where $\mathcal{E}_x$ is the electric field in V/m and J_x is the current density in A/m^2. Write the corresponding equations in 1D and 2D and determine the units of ρ_{1D}, $\rho_{2D} = \rho_S$, and ρ_{3D}.

Solution:

For 1D, there is no current density, just current, so we have:
$\mathcal{E}_x = \rho_{1D} I_x$
in terms of units, we can write:
V/m = (?) A

To make the units match, we must have: (?) = V/(A-m) = **Ohms/m**

For 2D, the current density is in A/m:

$\mathcal{E}_x = \rho_{2D} J_x$
in terms of units, we can write:

V/m = (?) A/m

To make the units match, we must have: (?) = V/(A) = **Ohms**

For 3D, the current density is in A/m^2:

$\mathcal{E}_x = \rho_{3D} J_x$

in terms of units, we can write:

V/m = (?) A/m^2

To make the units match, we must have: (?) = V-m/(A) = **Ohms-m**

ρ_{1D}	$\Omega - \mathrm{m}^{-1}$
ρ_{2D}	Ω
ρ_{3D}	$\Omega - \mathrm{m}$

ECE 656 Homework (Week 6) Solutions (continued)

7) The general expression for conductance,

$$G = \frac{2q^2}{h}\int \mathcal{T}(E)M(E)\left(-\frac{\partial f_0}{\partial E}\right)dE,$$

can be written as

$$G = \frac{2q^2}{h}\langle\langle \mathcal{T}(E)\rangle\rangle\langle M(E)\rangle.$$

For a 3D resistor in the diffusive limit,

$$G_{3D} = \frac{2q^2}{h}\langle\langle \lambda(E)\rangle\rangle\langle M(E)/A\rangle\frac{A}{L}.$$

Derive the general expressions for $\langle M(E)/A\rangle$ and for $\langle\langle \lambda(E)\rangle\rangle$ in terms of their energy-dependent quantities, $M(E)$ and $\lambda(E)$. HINT: Begin at the ballistic limit and determine $\langle M(E)/A\rangle$ first.

Solution:

Begin in the ballistic limit, where $\mathcal{T} = 1$ and we have:

$$G = \int_{-\infty}^{+\infty}\frac{2q^2}{h}M(E)\left(-\frac{\partial f_0}{\partial E}\right)dE = \frac{2q^2}{h}\langle M\rangle$$

$$\langle M\rangle = \int M(E)\left(-\frac{\partial f_0}{\partial E}\right)dE$$

Note that:

$$\int_{-\infty}^{+\infty}\left(-\frac{\partial f_0}{\partial E}\right)dE = -\int_{-\infty}^{+\infty}df_0 = -f_0(+\infty) + f_0(-\infty) = -0 + 1 = 1$$

so we can write:

$$\langle M\rangle = \frac{\int_{-\infty}^{+\infty}M(E)\left(-\frac{\partial f_0}{\partial E}\right)dE}{\int_{-\infty}^{+\infty}\left(-\frac{\partial f_0}{\partial E}\right)dE}. \qquad \text{(i)}$$

Equation (i) looks like an average. We interpret it as the average number of channels in the Fermi window.

ECE 656 Homework (Week 6) Solutions (continued)

Now include the transmission:

$$G = \frac{2q^2}{h}\int \mathcal{T}(E)\, M(E)\left(-\frac{\partial f_0}{\partial E}\right)dE$$

$$G = \frac{2q^2}{h}\frac{\int \mathcal{T}(E)\, M(E)\left(-\frac{\partial f_0}{\partial E}\right)dE}{\langle M\rangle}\langle M\rangle$$

so we define:

$$\langle\langle \mathcal{T}(E)\rangle\rangle = \frac{\int \mathcal{T}(E)\, M(E)\left(-\frac{\partial f_0}{\partial E}\right)dE}{\int M(E)\left(-\frac{\partial f_0}{\partial E}\right)dE}$$

In the diffusive limit:

$$\left\langle\left\langle \frac{\lambda(E)}{L}\right\rangle\right\rangle = \frac{\int \frac{\lambda(E)}{L} M(E)\left(-\frac{\partial f_0}{\partial E}\right)dE}{\int M(E)\left(-\frac{\partial f_0}{\partial E}\right)dE}$$

Multiply through by L to find

$$\langle\langle \lambda(E)\rangle\rangle = \frac{\int \lambda(E)\, M(E)\left(-\frac{\partial f_0}{\partial E}\right)dE}{\int M(E)\left(-\frac{\partial f_0}{\partial E}\right)dE} \qquad \text{(ii)}$$

Writing (i) and (ii) on a modes per unit area basis, we find:

$$\boxed{\begin{aligned}\langle M\rangle &= \frac{\int_{-\infty}^{+\infty} \left(M(E)/A\right)\left(-\frac{\partial f_0}{\partial E}\right)dE}{\int_{-\infty}^{+\infty}\left(-\frac{\partial f_0}{\partial E}\right)dE}\\ \langle\langle \lambda(E)\rangle\rangle &= \frac{\int \lambda(E)\left(M(E)/A\right)\left(-\frac{\partial f_0}{\partial E}\right)dE}{\int \left(M(E)/A\right)\left(-\frac{\partial f_0}{\partial E}\right)dE}\end{aligned}}$$

ECE 656 Homework (Week 7) Solutions (continued)

8) According to equ. (6.9) on p. 182 of *Advanced Semiconductor Fundamentals*, 2nd Ed., (R.F. Pierret, 2003) the mobility of electrons in bulk Si for $N_D = 10^{14}\ \text{cm}^{-3}$ at $T = 300$ K is 1268 $\text{cm}^2/\text{V-s}$ and at $N_D = 10^{20}\ \text{cm}^{-3}$ it is 95 $\text{cm}^2/\text{V-s}$. Assume an energy independent mean-free-path and **determine the mean-free-path**, λ_0 in both cases.

Solution:

The assumption is that we are dealing with a 3D semiconductor. The conductivity is written as:

$$\sigma = \frac{2q^2}{h}\lambda_0 \langle M_{3D}/A \rangle \qquad \text{(i)}$$

$$\langle M_{3D}/A \rangle = g_V \frac{m^* k_B T}{2\pi\hbar^2}\mathcal{F}_0(\eta_F) \qquad \text{(ii)}$$

What effective mass do we use in this expression?
Not the density of state effective mass.
Not the conductivity effective mass.
We should use the "distribution of modes" effective mass (See Jeong, et al., *J. Appl. Phys.*, **107**, 023707, 2010).

$$g_V m^* = m^*_{DOM} = 2m^*_t + 4\sqrt{m^*_t m^*_\ell} = 2.04 m_0 \qquad \text{(iii)}$$

Before we can use (ii), however, we must determine η_F.

$$n = N_C \mathcal{F}_{1/2}(\eta_F)$$

$N_C = 1.03 \times 10^{19}\ \text{cm}^{-3}$ (Pierret, Adv. Semiconductor Fundamentals, p. 113)

Note that the effective DOS makes use of the density-of-states effective mass:

$$m^*_{DOS} = 6^{2/3}\left(m_\ell m_t^2\right)^{1/3} = 1.06 m_0\ ,$$

which is quite different from the distribution of modes effective mass.

So now we have a **procedure:**

Given the carrier density, solve for η_F: $n_0 = N_C \mathcal{F}_{1/2}(\eta_F)$

1) Next, solve for the average number of channels in the Fermi window:

$$\langle M_{3D}/A \rangle = \frac{m^*_{DOM} k_B T}{2\pi\hbar^2}\mathcal{F}_0(\eta_F)$$

3) Then find the conductivity from the given data: $\sigma = n_0 q \mu_n$

4) Finally, solve for the mean-free-path:

$$\lambda_0 = \frac{\sigma}{(2q^2/h)}\frac{1}{\langle M_{3D}/A \rangle}$$

ECE 656 Homework (Week 6) Solutions (continued)

Case i): $n_0 = N_D = 10^{14}\ \text{cm}^{-3}$ (non-degenerate)

$$\eta_F = \ln\left(n_0/N_C\right) = -11.5$$
$$\left\langle M_{3D}/A\right\rangle = \frac{m^*_{DOM}k_BT}{2\pi\hbar^2}\mathcal{F}_0\left(-11.5\right) = 1.07\times 10^{8}\ \text{cm}^{-2}$$
$$\sigma = n_0 q\mu_n = 10^{14}\times 1.6\times 10^{-19}\times 1268 = 2.03\times 10^{-2}\ \text{S/cm}$$
$$\sigma = n_0 q\mu_n = 10^{14}\times 1.6\times 10^{-19}\times 1268 = 2.03\times 10^{-2}$$
$$\frac{\sigma}{\left(2q^2/h\right)} = 263$$
$$\lambda_0 = \frac{\sigma}{\left(2q^2/h\right)}\frac{1}{\left\langle M_{3D}/A\right\rangle} = 263\times\frac{1}{1.07\times 10^{8}} = 25\ \text{nm}$$
$$\boxed{\lambda_0 = 25\ \text{nm}}$$

Case ii): $n_0 = N_D = 10^{20}\ \text{cm}^{-3}$ (degenerate)

$$n_0 = N_C\mathcal{F}_{1/2}\left(\eta_F\right) \rightarrow 10^{20} = 1.03\times 10^{19}\mathcal{F}_{1/2}\left(\eta_F\right) \rightarrow \eta_F = 5.34$$

$$\left\langle M_{3D}/A\right\rangle = \frac{m^*_{DOM}k_BT}{2\pi\hbar^2}\mathcal{F}_0\left(5.34\right) = 1.1\times 10^{13}\ \mathcal{F}_0\left(5.34\right)\text{cm}^{-2} = 5.87\times 10^{13}\ \text{cm}^{-2}$$

$$\sigma = n_0 q\mu_n = 10^{20}\times 1.6\times 10^{-19}\times 95 = 1.52\times 10^{3}$$

$$\frac{\sigma}{\left(2q^2/h\right)} = 1.97\times 10^{7}$$

$$\lambda_0 = \frac{\sigma}{\left(2q^2/h\right)}\frac{1}{\left\langle M_{3D}/A\right\rangle} = 1.97\times 10^{7}\times\frac{1}{5.87\times 10^{13}} = 3.4\ \text{nm}$$

$$\boxed{\lambda_0 = 3.4\ \text{nm}}$$

ECE 656 Homework (Week 6) Solutions (continued)

9) For solving problems involving 2D electrons in parabolic band semiconductors (i.e. as in the channel of a transistor), we can use the result in Lundstrom and Jeong, Appendix, p. 218, eqn. (A.30)

$$\sigma_S = \frac{2q^2}{h}\lambda_0 \frac{\sqrt{2m^* k_B T}}{\pi\hbar}\Gamma\left(r+\frac{3}{2}\right)\mathcal{F}_{r-1/2}\left(\eta_F\right)$$

where *r* is the characteristic exponent for scattering,
$\lambda(E) = \lambda_0\left[\left(E-E_C\right)/k_B T\right]^r$.
Use this expression and answer the following two questions.

9a) Assume a constant MFP,
$\lambda(E) = \lambda_0$ (i.e. $r = 0$)

and **work out** an expression for the 2D mobility in terms of the MFP. Your results should be valid for any level of carrier degeneracy. **Simplify** your results for T = 0K and for non-degenerate conditions.

9b) Assume a "power law" MFP describe by
$\lambda(E-E_C) = \lambda_0\left[\left(E-E_C/k_B T\right)\right]^r$,

where "*r*" is a characteristic exponent that describes scattering. Repeat problem 4a) for this energy-dependent MFP. **Note:** for T = 0 K, only one energy matters, so it is best just to write $\lambda(E)$ and not use the power law form.

Solution:

9a) *r* = 0 energy independent MFP

$$\sigma_S = \frac{2q^2}{h}\lambda_0 \frac{\sqrt{2m^* k_B T}}{\pi\hbar}\Gamma\left(\frac{3}{2}\right)\mathcal{F}_{-1/2}\left(\eta_F\right) \equiv n_S q \mu_n \qquad \text{(i)}$$

recall that $n_S = \frac{m^* k_B T}{\pi\hbar^2}\mathcal{F}_0\left(\eta_F\right)$ (valley degeneracy of one is assumed)

$$\mu_n = \frac{1}{n_S}\frac{2q}{h}\lambda_0 \frac{\sqrt{2m^* k_B T}}{\pi\hbar}\Gamma\left(\frac{3}{2}\right)\mathcal{F}_{-1/2}\left(\eta_F\right) = \frac{\frac{2q}{h}\lambda_0 \frac{\sqrt{2m^* k_B T}}{\pi\hbar}\Gamma\left(\frac{3}{2}\right)\mathcal{F}_{-1/2}\left(\eta_F\right)}{\frac{m^* k_B T}{\pi\hbar^2}\mathcal{F}_0\left(\eta_F\right)}$$

ECE 656 Homework (Week 6) Solutions (continued)

$$\mu_n = \frac{\frac{q}{\pi}\lambda_0\sqrt{\frac{2}{m^* k_B T}}\,\Gamma\left(\frac{3}{2}\right)\mathcal{F}_{-1/2}(\eta_F)}{\mathcal{F}_0(\eta_F)} \qquad \Gamma\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{2}$$

$$\mu_n = \frac{q\lambda_0\sqrt{\frac{2k_BT}{\pi m^*}}}{2k_BT}\frac{\mathcal{F}_{-1/2}(\eta_F)}{\mathcal{F}_0(\eta_F)} = \frac{\lambda_0 \upsilon_T}{2(k_BT/q)}\frac{\mathcal{F}_{-1/2}(\eta_F)}{\mathcal{F}_0(\eta_F)}$$

The result is:

$$\boxed{\mu_n = \frac{(\lambda_0\upsilon_T/2)}{(k_BT/q)}\frac{\mathcal{F}_{-1/2}(\eta_F)}{\mathcal{F}_0(\eta_F)}}$$

For non-degenerate statistics, this simplifies to:

$$\boxed{\mu_n = \frac{(\lambda_0\upsilon_T/2)}{(k_BT/q)}}$$

For degenerate statistics, is it best to begin again with:

$$\sigma_S = \frac{2q^2}{h}M_{2D}(E_F)\lambda(E_F) = nq\mu_n$$

$$\mu_n = \frac{1}{n_S}\frac{2q}{h}M_{2D}(E_F)\lambda(E_F) = \frac{1}{m^*/(\pi\hbar^2)(E_F - E_C)}\frac{2q}{h}\frac{\sqrt{2m^*(E_F-E_C)}}{\pi\hbar}\lambda(E_F)$$

$$\mu_n = \frac{1}{(E_F-E_C)/q}\frac{\sqrt{2(E_F-E_C)/m^*}}{\pi}\lambda(E_F)$$

$$\mu_n = \frac{1}{(E_F-E_C)/q}\frac{(2\upsilon_F/\pi)\lambda_F}{2} \qquad (\lambda_F = \lambda(E_F))$$

$$\boxed{\mu_n = \frac{[(2\upsilon_F/\pi)\lambda_F]/2}{(E_F-E_C)/q}}$$

ECE 656 Homework (Week 6) Solutions (continued)

9b) energy dependent MFP

Use the result in Lundstrom and Jeong, Appendix, p. 218, eqn. (A.30)

$$\sigma_S = \frac{2q^2}{h}\lambda_0 \frac{\sqrt{2m^* k_B T}}{\pi\hbar}\Gamma(r+3/2)\mathcal{F}_{r-1/2}(\eta_F)$$

Repeating the derivation of part 2a), we find:

$$\mu_n = \frac{\frac{q}{\pi}\lambda_0\sqrt{\frac{2}{m^* k_B T}}\Gamma(r+3/2)}{} \frac{\mathcal{F}_{r-1/2}(\eta_F)}{\mathcal{F}_0(\eta_F)}$$

$$\mu_n = \frac{\lambda_0\sqrt{\frac{2k_B T}{\pi m^*}}\frac{\Gamma(r+3/2)}{\sqrt{\pi}}}{k_B T/q}\frac{\mathcal{F}_{r-1/2}(\eta_F)}{\mathcal{F}_0(\eta_F)}$$

$$\boxed{\mu_n = \frac{(\lambda_0 \upsilon_T/2)}{k_B T/q}\frac{\Gamma(r+3/2)}{\Gamma(3/2)}\frac{\mathcal{F}_{r-1/2}(\eta_F)}{\mathcal{F}_0(\eta_F)}}$$

For non-degenerate statistics, this simplifies to:

$$\boxed{\mu_n = \frac{(\lambda_0 \upsilon_T/2)}{k_B T/q}\frac{\Gamma(r+3/2)}{\Gamma(3/2)}}$$

For degenerate statistics, the answer is that same as in 4a).

$$\boxed{\mu_n = \frac{\left[(2\upsilon_F/\pi)\lambda_F\right]/2}{(E_F - E_C)/q}}$$

10) Assume an n-channel MOSFET at T = 300 K with $n_S = 10^{13}\ \text{cm}^{-3}$. Assume that only the lowest subband is occupied and compute $\langle M_{2D}\rangle$, the average number of modes in the Fermi window per micrometer of channel width.

Solution:

The first step is to determine the location of the Fermi level.

$$n_S = N_{2D}\mathcal{F}_0(\eta_F) = g_V \frac{m^* k_B T}{\pi\hbar^2}\ln\left(1+e^{\eta_F}\right) \qquad \text{(i)}$$

For the first unprimed subband,

$m^* = m_t = 0.19 m_0$

$g_V = 2$

ECE 656 Homework (Week 7) Solutions (continued)

$$N_{2D} = g_V \frac{m^* k_B T}{\pi \hbar^2} = 2\frac{0.19 \times 9.11 \times 10^{-31} \times 1.38 \times 10^{-23} \times 300}{3.14 \times \left(1.055 \times 10^{-34}\right)^2}$$

$$N_{2D} = 4.1 \times 10^{12} \text{ cm}^{-2}$$

$$n_S = N_{2D} \ln\left(1 + e^{\eta_F}\right) \rightarrow \eta_F = \ln\left(e^{n_S/N_{2D}} - 1\right) = \ln\left(e^{10/4.1} - 1\right)$$

$$\eta_F = 2.35$$

From problem 3):

$$\left\langle M \right\rangle = \frac{\int_{-\infty}^{+\infty} \left(M\left(E\right)/A\right)\left(-\frac{\partial f_0}{\partial E}\right) dE}{\int_{-\infty}^{+\infty} \left(-\frac{\partial f_0}{\partial E}\right) dE},$$

which can be worked out in 2D to find

$$\left\langle M_{2D} \right\rangle = g_V \frac{\sqrt{2m^* k_B T}}{\pi \hbar} \frac{\sqrt{\pi}}{2} \mathcal{F}_{-1/2}\left(\eta_F\right)$$

Putting in numbers:

$$\left\langle M_{2D} \right\rangle = 2\frac{\sqrt{2 \times 0.19 \times 9.11 \times 10^{-31} \times 1.38 \times 10^{-23} \times 300}}{3.14 \times 1.055 \times 10^{-34}} \frac{\sqrt{3.14}}{2} \mathcal{F}_{-1/2}\left(2.35\right)$$

$$\left\langle M_{2D} \right\rangle = 2.025 \times 10^8 \times \mathcal{F}_{-1/2}\left(2.35\right) = 2.025 \times 10^8 \times 1.607 = 3.25 \times 10^8 \text{ m}^{-1}$$

$$\boxed{\left\langle M_{2D} \right\rangle = 325 \ \mu\text{m}^{-1}}$$

Consider an L = 22 nm technology with W = L = 0.022 micrometers. We would have, $\left\langle M_{2D} \right\rangle = 7$, a fairly small number.

11) In 1D, we express the resistance of a long (diffusive) resistor by $R_{1D} = \left(1/\sigma_{1D}\right)L$. In 2D, we write $R_{2D} = \left(1/\sigma_{2D}\right)L/W$ and in 3D $R_{3D} = \left(1/\sigma_{3D}\right)L/A$. Assuming a degenerate conductor (i.e. T = 0 K), begin with $G_{ball} = \frac{2q^2}{h} M\left(E_F\right)$ and develop expressions for the 1D, 2D, and 3D "ballistic conductivities."

ECE 656 Homework (Week 7) Solutions (continued)

Solution:

$$G_{ball}^{1D} = \frac{2q^2}{h} M_{1D}\left(E_F\right) \equiv \sigma_{ball}^{1D} \frac{1}{L}$$

$$\sigma_{ball}^{1D} = \frac{2q^2}{h} M_{1D}\left(E_F\right) L$$

$$G_{ball}^{2D} = \frac{2q^2}{h} W M_{2D}\left(E_F\right) \equiv \sigma_{ball}^{2D} \frac{W}{L}$$

$$\sigma_{ball}^{2D} = \frac{2q^2}{h} M_{2D}\left(E_F\right) L$$

$$G_{ball}^{3D} = \frac{2q^2}{h} A M_{3D}\left(E_F\right) \equiv \sigma_{ball}^{3D} \frac{A}{L}$$

$$\sigma_{ball}^{3D} = \frac{2q^2}{h} M_{3D}\left(E_F\right) L$$

So the results are:

$$\boxed{\begin{aligned} \sigma_{ball}^{1D} &= \frac{2q^2}{h} M_{1D}\left(E_F\right) L \quad \left(\Omega / \mathrm{m}\right)^{-1} \\ \sigma_{ball}^{2D} &= \frac{2q^2}{h} M_{2D}\left(E_F\right) L \quad \left(\Omega\right)^{-1} \\ \sigma_{ball}^{3D} &= \frac{2q^2}{h} M_{3D}\left(E_F\right) L \quad \left(\Omega\text{-m}\right)^{-1} \end{aligned}}$$

To go further, we need to specify $M\left(E_F\right)$. Let's assume parabolic energy bands:

$$M_{\mathrm{1D}}\left(E_F\right) = H\left(E_F - E_C\right)$$

$$M_{\mathrm{2D}}\left(E_F\right) = \frac{\sqrt{2m^*\left(E_F - E_C\right)}}{\pi\hbar} H\left(E_F - E_C\right)$$

$$M_{\mathrm{3D}}\left(E_F\right) = A\frac{m^*}{2\pi\hbar^2}\left(E_F - E_C\right) H\left(E_F - E_C\right)$$

As an exercise, you might now want to derive the ballistic mobilities in 1D, 2D, and 3D.

ECE 656 Homework (Week 7) Solutions (continued)

12) One can derive a near-equilibrium current equation for a 2D, n-type conductor in the diffusive limit and write it as $J_n = \sigma_S \, d\left(F_n/q\right)/dx$ A/m. Derive the corresponding equation for a p-type semiconductor.

Solution:

$$I = \frac{2q}{h}\int_{-\infty}^{E_V} \mathcal{T}(E)\, M_V(E)\left(f_1 - f_2\right) dE$$

(the channels in the valence band are all below $E = E_V$.)

$$f_1 - f_2 \approx \left(-\frac{\partial f_1}{\partial E}\right) qV \qquad \mathcal{T}(E) = \frac{\lambda(E)}{\lambda(E)+L} \rightarrow \frac{\lambda(E)}{L} \quad \text{(diffusive transport)}$$

$$I = \left\{\frac{2q}{h}\int_{-\infty}^{E_V} \lambda(E)\, M_V(E)\left(-\frac{\partial f_0}{\partial E}\right) dE\right\}\frac{qV}{L}$$

$J_{px} = -I/W$ A/m (current density in 2D)

$qV = -\Delta F_p$

$$J_{px} = \left\{\frac{2q}{h}\int_{-\infty}^{E_V} \lambda(E)\left(M_V(E)/W\right)\left(-\frac{\partial f_0}{\partial E}\right) dE\right\}\frac{\Delta F_p}{L}$$

$$J_{px} = \left\{\frac{2q}{h}\int_{-\infty}^{E_V} \lambda(E)\left(M_V(E)/W\right)\left(-\frac{\partial f_0}{\partial E}\right) dE\right\}\frac{dF_p}{dx} = \sigma_{Sp}\frac{dF_p}{dx}$$

$$\boxed{\begin{aligned} J_{px} &= \sigma_{Sp}\frac{d\left(F_p/q\right)}{dx} \\ \sigma_{Sp} &= \frac{2q^2}{h}\int_{-\infty}^{E_V} \lambda(E)\left(M_V(E)/W\right)\left(-\frac{\partial f_0}{\partial E}\right) dE \end{aligned}}$$

ECE 656 Homework (Week 7) Solutions (continued)

13) Begin with $J_{nx} = \sigma_n d\left(F_n/q\right)/dx$ and derive the drift-diffusion equation for a 3D n-type semiconductor with parabolic energy bands. **Do not** assume Maxwell-Boltzmann statistics.

Solution:

Begin with:

$$J_{nx} = \sigma_n \frac{dF_n/q}{dx} \qquad \text{(i)}$$

$$n = N_C \mathcal{F}_{1/2}\left(\eta_F\right) \qquad \eta_F = \left(F_n - E_C\right)/k_B T \qquad N_C = \frac{1}{4}\left(\frac{2m^* k_B T}{\pi\hbar^2}\right)^{3/2}$$

Now find the gradient of the electrochemical potential:

$$\frac{dn}{dx} = N_C\left\{\frac{d}{d\eta_F}\mathcal{F}_{1/2}\left(\eta_F\right)\right\}\frac{d\eta_F}{dx} = N_C \mathcal{F}_{-1/2}\left(\eta_F\right)\left\{\frac{dF_n}{dx} - \frac{dE_C}{dx}\right\}\frac{1}{k_B T}$$

$$\frac{dF_n}{dx} = \frac{1}{N_C \mathcal{F}_{-1/2}\left(\eta_F\right)} k_B T \frac{dn}{dx} + \frac{dE_C}{dx} = \frac{\mathcal{F}_{1/2}\left(\eta_F\right)}{N_C \mathcal{F}_{1/2}\left(\eta_F\right)\mathcal{F}_{-1/2}\left(\eta_F\right)} k_B T \frac{dn}{dx} + \frac{dE_C}{dx}$$

$$\frac{d\left(F_n/q\right)}{dx} = \frac{\mathcal{F}_{1/2}\left(\eta_F\right)}{\mathcal{F}_{-1/2}\left(\eta_F\right)} \frac{k_B T}{q} \frac{1}{n} \frac{dn}{dx} + \frac{d\left(E_C/q\right)}{dx} \qquad \text{(ii)}$$

Insert (ii) in (i)

$$J_{nx} = \sigma_n \frac{dF_n/q}{dx} = \sigma_n\left\{\frac{\mathcal{F}_{1/2}\left(\eta_F\right)}{\mathcal{F}_{-1/2}\left(\eta_F\right)} \frac{k_B T}{q} \frac{1}{n} \frac{dn}{dx} + \mathcal{E}_x\right\}$$

$$J_{nx} = \sigma_n \mathcal{E}_x + \sigma_n \frac{\mathcal{F}_{1/2}\left(\eta_F\right)}{\mathcal{F}_{-1/2}\left(\eta_F\right)} \frac{k_B T}{q} \frac{1}{n} \frac{dn}{dx} \qquad \text{(iii)}$$

Now write

$$\sigma_n = nq\mu_n$$

and use in (iii) to find:

$$J_{nx} = nq\mu_n \mathcal{E}_x + \frac{\mathcal{F}_{1/2}\left(\eta_F\right)}{\mathcal{F}_{-1/2}\left(\eta_F\right)} k_B T \mu_n \frac{dn}{dx} \qquad \text{(iv)}$$

ECE 656 Homework (Week 7) Solutions (continued)

Define the diffusion coefficient as

$$D_n = \frac{\mathcal{F}_{1/2}(\eta_F)}{\mathcal{F}_{-1/2}(\eta_F)} \times \frac{k_B T}{q} \mu_n \qquad \text{(v)}$$

Note that for a nondegenerate semiconductor, $\eta_F = (F_n - E_C)/k_B T << 0$, and we would find

$$D_n = \frac{k_B T}{q} \mu_n ,$$

which is the familiar Einstein relation for non-degenerate semiconductors.

Finally, use (v) in (iv)

$$\boxed{\begin{aligned} J_{nx} &= nq\mu_n \mathcal{E}_x + qD_n \frac{dn}{dx} \\ \frac{D_n}{\mu_n} &= \frac{\mathcal{F}_{1/2}(\eta_F)}{\mathcal{F}_{-1/2}(\eta_F)} \frac{k_B T}{q} \end{aligned}}$$

14) When we write the resistance as $R = R_{\text{ball}}(1 + L/\lambda_0)$, we assume a constant (energy-independent) mean-free-path. What is the corresponding expression for an energy dependent mean-free-path, $\lambda(E)$? Is a plot of resistance vs. length of the resistor a straight line?

Solution:

Begin with the expression for the conductance:

$$G = \frac{2q^2}{h} \int \mathcal{T}(E) M(E) \left(-\frac{\partial f_0}{\partial E} \right) dE$$

$$G = \frac{2q^2}{h} \int \frac{\lambda(E)}{\lambda(E) + L} M(E) \left(-\frac{\partial f_0}{\partial E} \right) dE$$

$$R = \left(\frac{h}{2q^2} \right) \frac{1}{\int \frac{\lambda(E)}{\lambda(E)+L} M(E) \left(-\frac{\partial f_0}{\partial E} \right) dE} \qquad \text{(A)}$$

The corresponding ballistic resistance is:

$$R_{ball} = \left(\frac{h}{2q^2} \right) \frac{1}{\int M(E) \left(-\frac{\partial f_0}{\partial E} \right) dE}$$

ECE 656 Homework (Week 7) Solutions (continued)

So we can re-write the resistance by adding the ballistic resistance to the RHS and then subtracting it (in the square brackets below):

$$R = R_{ball} + \left(\frac{h}{2q^2}\right)\left[\frac{1}{\int \frac{\lambda(E)}{\lambda(E)+L} M(E)\left(-\frac{\partial f_0}{\partial E}\right) dE} - \frac{1}{\int M(E)\left(-\frac{\partial f_0}{\partial E}\right) dE}\right] \quad \text{(B)}$$

Now multiple and divide by $\int M(E)\left(-\frac{\partial f_0}{\partial E}\right) dE$ inside the square brackets:

$$R = R_{ball} + \left[\left\{\left(\frac{h}{2q^2}\right)\frac{1}{\int M(E)\left(-\frac{\partial f_0}{\partial E}\right) dE}\right\} \times \left\{\frac{\int M(E)\left(-\frac{\partial f_0}{\partial E}\right) dE}{\int \frac{\lambda(E)}{\lambda(E)+L} M(E)\left(-\frac{\partial f_0}{\partial E}\right) dE} - 1\right\}\right]$$

The first term inside the square brackets is just the ballistic resistance.

$$R = R_{ball} + \left[R_{ball} \times \left\{\frac{\int M(E)\left(-\frac{\partial f_0}{\partial E}\right) dE}{\int \frac{\lambda(E)}{\lambda(E)+L} M(E)\left(-\frac{\partial f_0}{\partial E}\right) dE} - 1\right\}\right]$$

$$R = R_{ball}\left[1 + \left\{\frac{\int M(E)\left(-\frac{\partial f_0}{\partial E}\right) dE}{\int \frac{\lambda(E)}{\lambda(E)+L} M(E)\left(-\frac{\partial f_0}{\partial E}\right) dE} - 1\right\}\right] \quad \text{(C)}$$

Now multiply the curly brackets by L and divide both terms inside the brackets by L.

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$$R = R_{ball}\left[1+\left\{\frac{\int M(E)\left(-\frac{\partial f_0}{\partial E}\right)dE}{\int \frac{\lambda(E)L}{\lambda(E)+L} M(E)\left(-\frac{\partial f_0}{\partial E}\right)dE} - \frac{1}{L}\right\}L\right]$$

which can be written as

$$R = R_{ball}\left[1+\left\{\frac{\int M(E)\left(-\frac{\partial f_0}{\partial E}\right)dE}{\int \left(\frac{1}{\lambda(E)}+\frac{1}{L}\right)^{-1} M(E)\left(-\frac{\partial f_0}{\partial E}\right)dE} - \frac{1}{L}\right\}L\right]$$

Now define an "apparent mean-free-path"

$$\frac{1}{\lambda_{app}(E)} = \frac{1}{\lambda(E)} + \frac{1}{L} \qquad \text{(D)}$$

so we can write the resistance as

$$R = R_{ball}\left[1+\left\{\frac{\int M(E)\left(-\frac{\partial f_0}{\partial E}\right)dE}{\int \lambda_{app}(E) M(E)\left(-\frac{\partial f_0}{\partial E}\right)dE} - \frac{1}{L}\right\}L\right]$$

Note that the apparent MFP is basically the smaller of the actual MFP and the length of the resistor. Now define the average apparent MFP:

$$\langle\langle \lambda_{app} \rangle\rangle = \frac{\int \lambda_{app}(E) M(E)\left(-\frac{\partial f_0}{\partial E}\right)dE}{\int M(E)\left(-\frac{\partial f_0}{\partial E}\right)dE} \qquad \text{(E)}$$

so the resistance vs. length is

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$$R = R_{ball}\left[1+\left\{\frac{1}{\langle\langle\lambda_{app}\rangle\rangle}-\frac{1}{L}\right\}L\right] \tag{F}$$

When the MFP is energy dependent, then the average mean-free-path depends on the length of the sample. This occurs because some energy channels and be ballistic and some diffusive and this changes as the sample length changes. Let's consider the constant MFP case, then we see

$$\langle\langle\lambda_{app}\rangle\rangle = \frac{\lambda_0 L}{\lambda_0 + 1}$$
$$\frac{1}{\langle\langle\lambda_{app}\rangle\rangle}-\frac{1}{L} = \lambda_0$$

and

$$R = R_{ball}\left[1+\left\{\frac{1}{\langle\langle\lambda_{app}\rangle\rangle}-\frac{1}{L}\right\}L\right] = R_{ball}\left[1+\frac{L}{\lambda_0}\right]$$

The resistance is linear with length, as expected.

Summary of final results:

$$\boxed{\begin{aligned}
&R = R_{ball}\left[1+\left\{\frac{1}{\langle\langle\lambda_{app}\rangle\rangle}-\frac{1}{L}\right\}L\right]\\
&\langle\langle\lambda_{app}\rangle\rangle = \frac{\int \lambda_{app}(E)M(E)\left(-\frac{\partial f_0}{\partial E}\right)dE}{\int M(E)\left(-\frac{\partial f_0}{\partial E}\right)dE}\\
&\frac{1}{\lambda_{app}(E)} \equiv \frac{1}{\lambda(E)}+\frac{1}{L}
\end{aligned}}$$

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Bottom line: The algebra is tedious, but the concept is easy to understand. When the MFP is energy dependent, then the transmission of each channel depends differently on energy, depending on its MFP and the sample length. The fraction of channels that are near-ballistic and near-diffusive changes with sample length, so the resistance does not scale linearly with length.

Because the apparent mean-free-path and average inverse apparent mean-free-path depend on sample length, L, a plot of R vs. L is not necessarily linear, when there is a strong energy dependence to the mean-free-path. This effect is more important for phonons, where the mean-free-paths very strongly with energy, than for electrons, where the mean-free-paths tend to vary more slowly with energy and for which only a few energy channels near the bottom of the ban are typically occupied.