SOLUTIONS: ECE 656 Homework (Week 7)

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1) This homework exercise will help you become familiar with how B –fields affect transport

Consider the equation of motion for an average electron,

$$\vec{F}_e = -q\vec{\mathcal{E}} - q\vec{\upsilon}\times\vec{B} = \frac{d\vec{p}}{dt} \quad . \tag{i}$$

Assume that the electron moves for a time, τ_m, then scatters, returning the average momentum to zero, so

$$\frac{dp}{dt} = -\frac{p}{\tau_m} \quad . \tag{ii}$$

Assuming that $\vec{p} = m^*\vec{\upsilon}$, we find an equation for the average velocity as

$$\vec{\upsilon} = -\frac{q\tau}{m^*}\vec{\mathcal{E}} - \frac{q\tau}{m^*}\vec{\upsilon}\times\vec{B} . \tag{iii}$$

This equation can be solved exactly for the velocity (see prob. 4.18, Lundstrom, *Fundamentals of Carrier Transport*, 2000), but let's take a different approach.

1a) Assume carriers move in 2D and that only a z-directed B-field is present. Evaluate eqn. (iii) and find two coupled equations for υ_x and υ_y .

Solution:

Assuming a z-directed B-field, it is straightforward to show from (iii) that

$$\begin{aligned} \upsilon_x &= -\frac{q\tau_m}{m^*}\mathcal{E}_x - \frac{q\tau_m}{m^*}\upsilon_y B_z \\ \upsilon_y &= -\frac{q\tau_m}{m^*}\mathcal{E}_y + \frac{q\tau_m}{m^*}\upsilon_x B_z \end{aligned} \tag{*}$$

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1b) Solve the two equations for υ_x and υ_y in terms of the electric field and the B-field.

Solution:

From (*), it is straightforward to show:

$$\upsilon_x = \frac{-\mu_n \mathcal{E}_x + \mu_n^2 \mathcal{E}_y B_z}{1+(\omega_c \tau)^2}$$
$$\upsilon_y = \frac{-\mu_n \mathcal{E}_y - \mu_n^2 \mathcal{E}_x B_z}{1+(\omega_c \tau)^2} \qquad (**)$$

Where $\omega_c = \frac{qB_z}{m^*}$ is the so-called cyclotron frequency.

1c) Write the current densities as

$$J_x = -n_S q \upsilon_x \qquad \text{(iva)}$$
$$J_y = -n_S q \upsilon_y \qquad \text{(ivb)}$$

and use the results of problem 5b) to find the current densities as

$$J_x = -nq\upsilon_x = \frac{\sigma_n}{1+(\mu_n B_z)^2}\left(\mathcal{E}_x - \mu_n B_z \mathcal{E}_y\right) \qquad \text{(va)}$$
$$J_y = -nq\upsilon_y = \frac{\sigma_n}{1+(\mu_n B_z)^2}\left(\mathcal{E}_y + \mu_n B_z \mathcal{E}_x\right), \qquad \text{(vb)}$$

which can also be written as

$$\begin{pmatrix} J_x \\ J_y \end{pmatrix} = \frac{\sigma_n}{1+(\mu_n B_z)^2}\begin{bmatrix} 1 & -\mu_n B_z \\ \mu_n B_z & 1 \end{bmatrix}\begin{pmatrix} \mathcal{E}_x \\ \mathcal{E}_y \end{pmatrix} \qquad \text{(via)}$$

or as

$$J_i = \sigma_{ij}(B_z)\mathcal{E}_j. \qquad \text{(vib)}$$

Note that the magnetic field affects both the diagonal and off-diagonal components of the magnetoconductivity tensor. **Explain** why there is no Hall factor, r_H, in the result.

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Solution:

These results follow directly from (**). Note that we are considering the motion of an **average** electron with a momentum relaxation time of τ_m. We are neglecting the variation of scattering time with energy and considering only an average case.

Consequently, $\langle \tau_m \rangle = \tau_m$ and $\langle \tau_m^2 \rangle = \tau_m^2$ and the Hall factor is one. The algebra in this approach is simpler, but we are missing the fact that there is a Hall factor.

1d) Solve eqn. (via) for the electric field and show that

$$\begin{pmatrix} \mathcal{E}_x \\ \mathcal{E}_y \end{pmatrix} = \frac{1}{\sigma_n} \begin{bmatrix} 1 & \mu_n B_z \\ -\mu_n B_z & 1 \end{bmatrix} \begin{pmatrix} J_x \\ J_y \end{pmatrix} \tag{vii}$$

According to eqn. (vii), the longitudinal magnetoresistivity is independent of the *B*-field (while the longitudinal magnetoconductivity depends on *B* as shown in eqn. (via). Equation (vii) shows that the Hall voltage is proportional to *B*.

Solution:

We can re-write (vi) as

$$\begin{pmatrix} J_x \\ J_y \end{pmatrix} = \begin{bmatrix} \sigma_L & -\sigma_T \\ \sigma_T & \sigma_L \end{bmatrix} \begin{pmatrix} \mathcal{E}_x \\ \mathcal{E}_y \end{pmatrix} \tag{***}$$

where

$$\sigma_L = \frac{\sigma_n}{1+\left(\mu_n B_z\right)^2} \qquad \sigma_T = \frac{\sigma_n \mu_n B_z}{1+\left(\mu_n B_z\right)^2}$$

If we invert (***), we find

$$\begin{pmatrix} \mathcal{E}_x \\ \mathcal{E}_y \end{pmatrix} = \begin{bmatrix} \rho_L & \rho_T \\ -\rho_T & \rho_L \end{bmatrix} \begin{pmatrix} J_x \\ J_y \end{pmatrix} \tag{****}$$

where

$$\rho_L = \frac{\sigma_L}{\sigma_L^2 + \sigma_T^2} = \frac{1}{\sigma_n} \qquad \rho_T = \frac{\sigma_T}{\sigma_L^2 + \sigma_T^2} = \frac{\mu_n B_z}{\sigma_n}$$

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The final result,

$$\boxed{\begin{pmatrix} \mathcal{E}_x \\ \mathcal{E}_y \end{pmatrix} = \frac{1}{\sigma_n}\begin{bmatrix} 1 & \mu_n B_z \\ -\mu_n B_z & 1 \end{bmatrix}\begin{pmatrix} J_x \\ J_y \end{pmatrix}}$$

shows that the longitudinal magneto**resistivity** is independent of B while the longitudinal magneto**conductivity** (from (via)) depends on the B-field. Comparing to prob. 5), we see that we get the same result for the magnetoresistivity without assuming a small B-field.

1e) Show that for small B-fields, eqn. (via) can be written as

$$\vec{J}_n = \sigma_n \vec{\mathcal{E}} - (\sigma_n \mu_n)\vec{\mathcal{E}} \times \vec{B} \qquad \text{(vii)}$$

Note that while this analysis is simpler than solving the BTE, by beginning with an average electron with an average momentum, $\vec{p}$, we have missed the averaging of the distribution of momenta which leads to a non-unity Hall factor, r_H.

Solution:

Simply expand out (vii) and show that it gives (via) for small *B*.

$$\boxed{\begin{aligned} \begin{pmatrix} J_x \\ J_y \end{pmatrix} &= \frac{\sigma_n}{1+(\mu_n B_z)^2}\begin{bmatrix} 1 & -\mu_n B_z \\ \mu_n B_z & 1 \end{bmatrix}\begin{pmatrix} \mathcal{E}_x \\ \mathcal{E}_y \end{pmatrix} \\ &\approx \sigma_n \begin{bmatrix} 1 & -\mu_n B_z \\ \mu_n B_z & 1 \end{bmatrix}\begin{pmatrix} \mathcal{E}_x \\ \mathcal{E}_y \end{pmatrix} \\ &= \sigma_n \vec{\mathcal{E}} - (\sigma_n \mu_n)\vec{\mathcal{E}} \times \vec{B} \end{aligned}}$$

2) Hall factors are important to consider when analyzing experiments. Answer the following questions.

2a) Assuming parabolic energy bands, derive an expression for the Hall factor in 3D and show that for ionized impurity scattering, it gives $r_H = 1.93$.

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Solution:

The definition of the Hall factor is:

$$r_H \equiv \langle\langle \tau_m^2 \rangle\rangle \big/ \langle\langle \tau_m \rangle\rangle^2 \tag{i}$$

Assume power law scattering:

$$\tau_m = \tau_0 \left(E/k_B T_L \right)^s$$

Recall that the average scattering time is:

$$\langle\langle \tau_m \rangle\rangle = \tau_0 \frac{\Gamma(s+5/2)}{\Gamma(5/2)}$$

Note also that

$$\tau_m^2 = \tau_0^2 \left(E/k_B T_L \right)^{2s}$$

is in power law form with a characteristic exponent of 2s instead of s, so

$$\langle\langle \tau_m^2 \rangle\rangle = \tau_0^2 \frac{\Gamma(2s+5/2)}{\Gamma(5/2)}$$

and

$$r_H = \frac{\Gamma(2s+5/2)\Gamma(5/2)}{\left[\Gamma(s+5/2)\right]^2} = 1.93$$

Now using (i), we find

$$r_H = \frac{\Gamma(2s+5/2)\Gamma(5/2)}{\left[\Gamma(s+5/2)\right]^2}$$

Assuming $s = 3/2$ for II scattering, we find

$$r_H = \frac{\Gamma(11/2)\Gamma(5/2)}{\left[\Gamma(4)\right]^2}$$

Recall some properties of the Gamma function:

$\Gamma(n) = (n-1)!$ (when n is an integer)

$\Gamma(1/2) = \sqrt{\pi}$

$\Gamma(p+1) = p\Gamma(p)$

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Accordingly, we find:

$$\Gamma(4) = (3)! = 6$$

$$\Gamma(11/2) = \frac{9}{2}\Gamma(9/2) = \frac{9}{2}\frac{7}{2}\Gamma(7/2) = \frac{9}{2}\frac{7}{2}\frac{5}{2}\Gamma(5/2)$$

$$\Gamma(5/2) = \frac{3}{2}\Gamma(3/2) = \frac{3}{2}\frac{1}{2}\Gamma(1/2) = \frac{3}{4}\sqrt{\pi}$$

Putting it all together:

$$r_H = \frac{\Gamma(11/2)\Gamma(5/2)}{\left[\Gamma(4)\right]^2} = \frac{\frac{9}{2}\frac{7}{2}\frac{5}{2}\frac{3}{4}\sqrt{\pi}\,\frac{3}{4}\sqrt{\pi}}{6\times 6} = \frac{22\pi}{36} = 1.9828$$

$$\boxed{r_H = 1.9828}$$

2b) Assume parabolic energy bands and develop an expression for the Hall factor in 2D.

Solution:

Recall that for 2D, parabolic bands, non-degenerate conditions

$$\langle\langle \tau_m \rangle\rangle = \tau_0 \frac{\Gamma(s+2)}{\Gamma(2)}$$

As in 7a), we find

$$\langle\langle \tau_m^2 \rangle\rangle = \tau_0^2 \frac{\Gamma(2s+2)}{\Gamma(2)}$$

So the Hall factor is

$$r_H \equiv \frac{\langle\langle \tau_m^2 \rangle\rangle}{\langle\langle \tau_m \rangle\rangle^2} = \frac{\dfrac{\Gamma(2s+2)}{\Gamma(2)}}{\left(\dfrac{\Gamma(s+2)}{\Gamma(2)}\right)^2} = \frac{\Gamma(2s+2)\Gamma(2)}{\Gamma(s+2)^2} \qquad \boxed{r_H = \frac{\Gamma(2s+2)\Gamma(2)}{\Gamma(s+2)^2}}$$

2c) Ionized impurity scattering in graphene is said to vary as *E*. What is the Hall factor for graphene?

Solution:

We cannot use the results from 6b), because the energy bands in graphene are not parabolic **and** because graphene is highly degenerate. Under degenerate conditions, only the scattering time at the Fermi level matters, so

$$\langle\langle \tau_m \rangle\rangle = \tau_m(E_F)$$

$$\langle\langle \tau_m^2 \rangle\rangle = \{\tau_m(E_F)\}^2$$

Accordingly,

$$\boxed{r_H = \langle\langle \tau_m^2 \rangle\rangle \Big/ \langle\langle \tau_m \rangle\rangle^2 = 1}$$

Independent of the type of scattering.