34. SCALING THEORY OF DESIGN OF EXPERIMENTS

34.1 Review/Background:

There are six lectures in data series. Once we have the data, we have to analyze it. The key message is: ‘treat your data with respect’. So far three lectures are discussed. Once we have the data, prior to fitting or massaging we have some other tasks to do. We plot the data in a stem and leaf histogram, and exclude the outliers while comparing against theoretical candidate distribution. In lecture 32, various types of distribution are discussed. Distributions are classified as single parameter distributions, such as Poisson distribution, two parameter distribution, e.g., normal, lognormal, Weibull, exponential, Fermi-dirac, Bose-Einstein distributions, and three parameter distribution such as Pearson distribution. Afterward, we fit the data with distribution using Fisher’s maximum likelihood estimator. We test goodness of fit using Residual, Q-Q method, Cox-Oakes measure, Kolmogorov-Smirnov algorithm, Pearson chi-square test etc.

In this chapter, we will go back and discuss the problem of data generation to being with. We have to design the experiment in such a way so that we will do fewest numbers of experiments to get maximum number of information. In this lecture, we will study so called scaling theory of design of experiment. The next lecture, we will discuss the statistical theory of design of experiment.

Buckingham π theorem is the key theorem in dimensional analysis for modeling physical phenomena. Although the first work was accomplished by others, this theorem is formally described in 1914 by E. Buckingham [34.1]. Many people in nineteenth century, who worked on sound [34.2], electromagnetic, fluid flow [34.3], [34.4], like Newton, Rayleigh [34.2], used some variant of this theory.
34.2 Problem definition

Suppose we have a reliability problem like HCI, NBTI etc. and R is the rate of degradation. The theory is so complicated that we do not know anything like any differential equation, reaction-diffusion model, channel hot carrier impact-ionization etc. We only could guess that the degradation rate depends on temperature \((T)\), energy barrier height \((E_B)\) of the materials, e.g., Si-H, SiO\(_2\), or HfO\(_2\) bond breaking barrier height, Boltzmann’s constant \((k_B)\) and Planck’s constant \((\hbar)\). We are tabulating these things that these things could be important but we are not sure.

\[
R = f(T, E_B, k_B, \hbar)
\]  

34.1

We can get the same information with fewer numbers of experiments. Equation 34.1 can be converted in a form

\[
\frac{R}{(k_B T / \hbar)} = f\left(\frac{E_B}{k_B T}\right)
\]

34.2

It doesn’t matter what \(E_B\), \(k_B\), or \(T\) is individually. It depends on the ratio \(E_B / k_B T\). Similarly, on the left side of equation 34.2 the rate itself does not matter, what matters is the ratio \(R / (k_B T / \hbar)\). So, from equation 34.2, by doing only 10 experiments we can get same information as original 100. This is sort of rotating the axis in some way or collecting the variables so that they change together.

We don’t know the functional dependence. So, if we do the experiment for two variables like \((T\) and \(E_B\)) and 10 experiments for each, then you have to do 100 experiments, for three variables 1000 etc. We are going to discuss the basic things here. Suppose hot carrier degradation will depend on channel doping, halo implant, properties of side walls, and many other things. We will be doing forever the testing without having any results, because we will not be able to conclude.

34.3 Buckingham II theorem

Let’s assume that a function \(g\) depends on parameters \(q_1, q_2, q_3, \ldots, q_n\) such that
In last example, we discussed in problem definition section, the total number of parameters \( n \) was 4. Here, \( g \) could be a differential equation like

\[
g(q_1, q_2, q_3, \ldots, q_n) = 0
\]

Or, it could be an unknown black box with control parameters \( q_1, q_2, q_3, \ldots, q_n \). The same expression 34.3 can be expressed in terms of independent dimensionless ratio or \( \Pi \) parameters [34.5].

\[
G(\Pi_1, \Pi_2, \Pi_3, \ldots, \Pi_{n-m}) = 0
\]

Here, \( m \) is the minimum number of independent dimension given by \( r \), where \( r \) is the rank of the matrix.

### 34.4 Determination of the dimensionless variable \( \Pi \)

Exploiting the fundamental physical units (e.g. mass (\( M \)), length (\( L \)), time (\( t \)), temperature (\( \Theta \)), etc) and dimensional variables (e.g. temperature (\( T \)), energy barrier height (\( E_B \)), velocity \( V \), density of the fluid is \( \rho \), viscosity \( \mu \), etc), we have to form the dimensional matrix \( A \)

\[
A = \begin{bmatrix} P & R \\ Q & S \end{bmatrix}
\]

Here, \( P \) is a \( r \times r \) nonsingular matrix. From the \( A \) matrix we have to form exponent matrix.
Each row of the exponent matrix will act like exponent of the dimensional variables sequentially for a particular \( \Pi \) variable through the expression

\[
\Pi_i = q_1^{e_{i1}} q_2^{e_{i2}} \ldots \ldots q_N^{e_{iN}}
\]

As an example, for NBTI or HCI degradation rate we create dimensionless variables combining dimensional variables exploiting Buckingham \( \Pi \) theorem. Equation 34.1 as a function \( g \) of dimensional variables can be written as

\[
g(R,T,E_B,k_B,h) = 0
\]

These variables can be expressed through fundamental physical units mass, length, time, and temperature with corresponding exponents \( a, b, c, \) and \( d \). The representation looks like

\[
\text{Variable} \rightarrow M^a \times L^b \times t^c \times \Theta^d
\]

So, the final representation of the dimensional variables through fundamental physical units is

\[
\begin{align*}
E_B & \rightarrow M^1 \times L^2 \times t^{-2} \times \Theta^0 \\
T & \rightarrow M^0 \times L^0 \times t^0 \times \Theta^1 \\
R & \rightarrow M^0 \times L^0 \times t^{-1} \times \Theta^0 \\
k_B & \rightarrow M^1 \times L^2 \times t^{-2} \times \Theta^{-1} \\
h & \rightarrow M^1 \times L^2 \times t^{-1} \times \Theta^0
\end{align*}
\]

The exponents of the fundamental physical units will lead to the formation of the dimensional matrix.
Number of unknown of the matrix \( A \) is \( n = 5 \). Rank of the matrix is \( r = 3 \) (the rank or number of independent rows of the matrix can easily calculated using MATLAB function rank). The rank is also known as number of repeating variables. The number of independent dimensionless ratio or \( \Pi \) parameters is \( n - r = 2 \).

Depending on the rank 3 of the matrix, \( P \) will be 3-by-3 matrix. The \( P \) matrix from the left top of \( A \) will be singular. So, we picked up from the right top instead. Any 3-by-3 nonsingular matrix as \( P \) is fine. \( Q \) will be the rest of the rows from the same columns. So, \( P \) and \( Q \) matrices look like

\[
A = \begin{bmatrix}
M & L & t & \Theta \\
0 & 0 & 0 & 1 \\
1 & 2 & -2 & -1 \\
1 & 2 & -1 & 0 \\
0 & 0 & -1 & 0 \\
1 & 2 & -2 & 0 \\
\end{bmatrix} 
\]

\[\begin{array}{c}
k_B \\
h \\
R \\
E_B \\
\end{array}\]

The exponent matrix will be formed by multiplying the negative of \( Q \) with the inverse of \( P \) i.e., \( (P^{-1}) \) and adding two new column of identity matrix \( (I) \) to the right side. The variables of the exponent in the \( E \) matrix will be placed sequentially as variable of \( P \) and then \( Q \).
The elements of the exponent matrix are the power exponent of the corresponding variables. Dimensionless variables $\Pi_1$ and $\Pi_2$ correspond to first and second rows of the $E$ matrix. So, the final expression of the dimensionless parameters

$$\Pi_1 = \frac{\hbar R}{k_B T} \quad \text{and} \quad \Pi_2 = \frac{E_B}{k_B T}$$

Putting the units of the individual dimensional variables to the right sides, we can verify that dimensionless parameters $\Pi$ are unitless. So, the new function as a representative of the old one with new dimensionless variables will be

$$G \left( \Pi_1 = \frac{\hbar R}{k_B T}, \quad \Pi_2 = \frac{E_B}{k_B T} \right) = 0$$

Considering $E_B/k_B T$ as only input and $\hbar R/k_B T$ as the response another functional representation is

$$\frac{\hbar R}{k_B T} = f \left( \frac{E_B}{k_B T} \right)$$

If we did not know about $\hbar$ (people did not know about $\hbar$ before the invention of quantum mechanics), then

$$\frac{cR}{k_B T} = f \left( \frac{E_B}{k_B T} \right)$$
We said any 3-by-3 nonsingular matrix as \( P \) is fine and \( Q \) will be the rest of the rows from the same columns. Picking another nonsingular matrix as \( P \) from the dimensional matrix \( A \),

\[
P_2 = \begin{bmatrix} L & t & \Theta \\ 2 & -2 & -1 \\ 2 & -1 & 0 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} T \\ k_B \\ R \end{bmatrix}
\]

\[
Q_2 = \begin{bmatrix} L & t & \Theta \\ 2 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} E_B \\ T \end{bmatrix}
\]

The new exponent matrix will be

\[
E = [-QP^{-1}, I] = \begin{bmatrix} k_B & h & R & E_B & T \\ 0 & -1 & -1 & 1 & 0 \\ 1 & -1 & -1 & 0 & 1 \end{bmatrix}
\]

The variables of the columns of the exponent matrix change their sequence following the variables of \( P_2 \) and then \( Q_2 \). So, the new dimensionless parameters

\[
\Pi_1 = \frac{E_B}{hR} \quad and \quad \Pi_2 = \frac{k_B T}{hR}
\]

The ratio of the dimensionless variable will also be a dimensionless variable.

\[
\Pi_3 = \frac{\Pi_1}{\Pi_2} = \frac{E_B}{k_B T}
\]

The new function of the dimensionless variable will be
Considering \( \frac{k_B T}{\hbar R} \) as only input and \( \frac{E_B}{k_B T} \) as the response another functional representation is

\[
\frac{k_B T}{\hbar R} = f_2 \left( \frac{E_B}{k_B T} \right)
\]

Whatever be the \( T, E_B, \) or anything, we will forget about them, we have to consider only the dimensionless variables.

![Figure 34.1](image)

Figure. 34.1 plot among two dimensionless variables \( \frac{\hbar R}{k_B T} \) vs \( \frac{E_B}{k_B T} \). Here, we can represent it in 1D plot, instead of 2D plot \( (E_B, T, \) and \( R) \). Visualization and analysis of the results are much easier, and less effort is required to perform the experiment.

### 34.5 Recall the scaling theory of NBTI, HCl, and TDDB

We are recalling the reaction-diffusion model for NBTI and HCl. Through these models we explain the scaling theory of design of experiment. Since NBTI and HCl are
explained in previous chapters, instead of explaining anything again in details, we are putting the equations only. The reaction equation is

$$\frac{dN_{IT}}{dt} = k_f [N_0 - N_{IT}] - k_r N_{IT} N_H(0)$$  \hspace{1cm} 34.27

In steady-state condition, $N_0 \gg N_{IT}$ the reaction equation will look like

$$\frac{k_f N_0}{k_r} = N_{IT} N_H(0)$$  \hspace{1cm} 34.28

Generation of the interface trap will be the total number of hydrogen inside oxide

$$N_{IT}(t) \approx \int N_H(r, t) \, dV$$  \hspace{1cm} 34.29

Figure. 34.2 (left) diffusion mechanism of Hydrogen through the oxide due to NBTI (1D). (Right) diffusion mechanism of Hydrogen through the oxide due to HCI (2D).

In NBTI because of the 1D flow of the Hydrogen from the Si/SiO₂ interface, interface trap generation will follow power law of time with exponent 1/4. This 1/4 will come from function and not from Buckingham $\pi$ theorem.

$$N_{IT}^{NBTI}(t) = N_H(0) \times \sqrt{D_H t}$$  \hspace{1cm} 34.30
\[
N_{IT}(t) = \sqrt[4]{\frac{k_f N_0}{k_r}} (D_H t)^{1/4}
\]

In HCI because of the 2D flow of the Hydrogen from one source point, interface trap generation will follow power law of time with exponent 1/2.

\[
N_{IT}^{HCI}(t) = \left(\frac{\pi}{12}\right) N_H(0) \times (\sqrt{D_H t})^2
\]

\[
N_{IT}(t) = \sqrt[4]{\frac{k_f N_0}{k_r}} (D_H t)^{1/2}
\]

Figure 34.3 Charge pumping current vs. time with different bias condition. We have to do the experiment for the variable \(t_0\), instead of doing experiment for individual variables like \(k_f\), \(k_r\), and \(N_0\).

Irrespective of NBTI and HCI, the trap generation expression can be generalized as
\[ N^{SIH}_{IT} = \left( \frac{k_f (V_G, V_D) N_0}{k_r} \right)^\alpha \times t^n \]  

At the end of the day, the equation 34.34 can be rewritten as

\[ N^{SIH}_{IT} = \left( \frac{t}{t_0} \right)^n = f^{SIH}_{SIH} \left( \frac{t}{t_0} \right) \]

The function \( f^{SIH}_{SIH} \) has to come from experiment. Here, \( t_0 \) is the scaling variable, can be written as

\[ t_0(V_G, V_D) = g \left( \frac{k_f N_0}{k_r} \right) \]

It does not matter what are the \( k_f, k_r, \) and \( N_0 \). So, long the combination is the same, the result has to be the same. Therefore, we do not have to do so many experiments.

### 34.6 Application of dimensional analysis

Dimensional analysis is widely used in fluid dynamics (Rayleigh [34.6], Reynolds, Prandtl numbers), percolation theory, reliability problems, system neurobiology [34.7] etc. Newton predicted bending of light by gravitational field simply by dimensional analysis. He was off by a factor of 2 compared to Einstein.

Engineering models used to analyze the fluid dynamics through calculation or computer simulation are not reliable. Similitude is a concept applicable for the testing of engineering models. Similitude explains why wind-tunnels work. And why Wright brothers succeeded, why others failed. Next we discuss one example. There are so many variables involved that it will span a long period to complete the experiment testing are variables.
A drag force on a ball placed in fluid is $F$, fluid is moving with velocity $V$, the diameter of the ball is $D$, the density of the fluid is $\rho$ and viscosity $\mu$. The drag force can be represented as

$$F = f(D, V, \rho, \mu) \tag{34.37}$$

Equation 34.37 as a function $g$ of dimensional variables can be written as

$$g(F, D, V, \rho, \mu) = 0 \tag{34.38}$$

This problem has five variables and three fundamental units. Invoking Buckingham Π theorem, the system can be described by two dimensionless parameters. These variables can be expressed through fundamental physical units like mass (M), length (L), and time (t) with corresponding exponents $a$, $b$, and $c$.

$$Variable \rightarrow M^a \times L^b \times t^c \tag{34.39}$$

The dimensional variables through fundamental physical units are

- $D \rightarrow M^0 \times L^1 \times t^0$ (meter)
- $V \rightarrow M^0 \times L^1 \times t^{-1}$ (meter/sec)
- $\rho \rightarrow M^1 \times L^{-3} \times t^0$ (kg/meter$^{-3}$)
- $F \rightarrow M^1 \times L^1 \times t^{-2}$ (kg – meter/sec$^{-2}$)
- $\mu \rightarrow M^1 \times L^{-1} \times t^{-1}$ (kg/meter/sec)

The exponents of the fundamental physical units will lead to the formation of the dimensional matrix
Similar to previous examples we calculate parameter $s$. Number of unknown of the matrix $A$ is $n = 5$. Rank of the matrix is $r = 3$ (the rank or number of independent rows of the matrix can easily calculated using MATLAB function rank). The number of independent dimensionless ratio or $\Pi$ parameters is $n - r = 2$.

Since the rank of the matrix is 3 of the matrix, any 3-by-3 nonsingular matrix as $P$ is fine. $Q$ will be the rest of the rows from the same columns. So, $P$ and $Q$ matrices look like

$$
P = \begin{bmatrix} M & L & t & D \\
0 & 1 & 0 & V \\
0 & 1 & -1 & \rho \\
1 & -3 & 0 & \mu \\
1 & 1 & -2 & F \\
-1 & -1 & -1 & \mu \end{bmatrix}
$$

$$
Q = \begin{bmatrix} M & L & t & F \\
1 & 1 & -2 & \mu \\
1 & -1 & -1 & \mu \end{bmatrix}
$$

The exponent matrix will look like

$$
E = [-QP^{-1}, I] = \begin{bmatrix} D & V & \rho & F & \mu \\
-2 & -2 & -1 & 1 & 0 \\
-1 & 1 & -1 & 0 & 1 \end{bmatrix}
$$

So, the final expression of the dimensionless parameters

$$
\Pi_1 = \frac{F}{\rho V^2 D^2} \text{ and } \Pi_2 = \frac{\mu}{\rho V D}
$$

Putting the units of the individual dimensional variables to the right sides, we can verify that dimensionless parameters $\Pi$ are unitless. So,
\[
G \left( \Pi_1 = \frac{F}{\rho V^2 D^2}, \quad \Pi_2 = \frac{\mu}{\rho V D} \right) = 0 \quad 34.46
\]

\[
\frac{F}{\rho V^2 D^2} = f \left( \frac{\mu}{\rho V D} \right) \quad 34.47
\]

If we add extra variables which are unimportant- they will either disappear or appear as normalized variables that will be shown to be irrelevant experimentally. Dimensional analysis is related to the principle component analysis in an interesting way (e.g. ‘Recommended for you’ by Amazon and Netflix) [34.8].

### 34.7 Conclusion:

Scaling of variables is a very important way of reducing the number of variables in an experiment. However, scaling requires that we have some idea about the key variables of the problem. There are many applications of the scaling theory, especially in fluid mechanics. The problem is complex, similar to that of reliability, and therefore scaling provides enormous simplification [34.9]. Some of the problems may not be fully specified in terms of explicitly stated variables. If we have multiple variables with same dimension like channel length and gate oxide thickness, or channel doping and halo implant, the Buckingham \( \Pi \) theorem will not be able to separate them apart, therefore we need additional methods. The Fisher/Taguchi method helps design those experiments.

### References


[34.9] The reliability example I used is from a book chapter on “Some Unifying Concepts in Reliability Physics, Mathematical Models, and Statistics” by R. E. Thomas.