Lecture 18
Analytical approaches

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Key results from last time

- Cantilevers have multiple eigenmodes
- The resonance frequencies are in specific ratios
- Each resonance peak you chose for operation has its own effective k
- Energy dissipation and virial

$$\text{If } \omega = \omega_0 \text{ then}$$

$$\sin(\phi) = \left\{ \frac{A}{A_0} - \frac{Q}{\pi k A A_0} E_{diss} \right\}$$

$$\cos(\phi) = -\frac{2Q}{kA^2} \langle F_{ts} \cdot x \rangle$$

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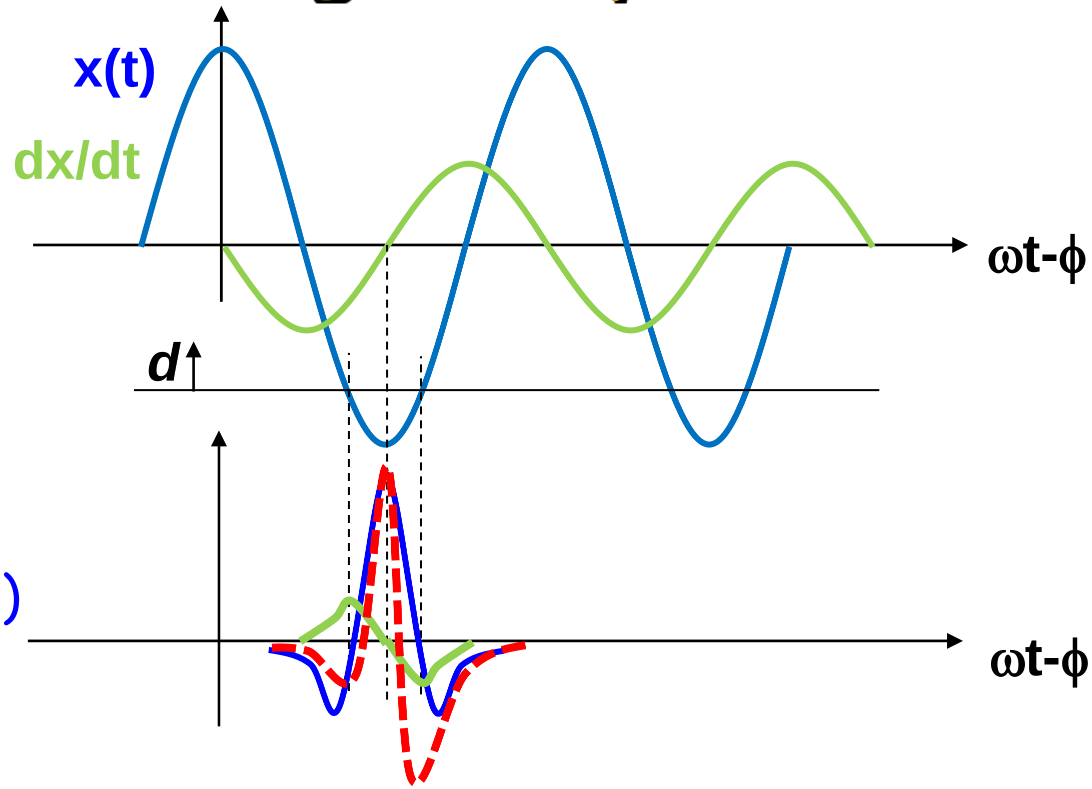
Understanding dissipation and virial

$$x(t) = A\cos(\omega t - \phi)$$



Fts (conservative)

Fts (dissipative)

$$F_{ts} = F_{ts}^{dissipative} + F_{ts}^{conservative}$$

$$E_{diss} = \int_{t=0}^{2\pi/\omega} \dot{x} \times F_{ts}(d,\dot{d})dt = \int_{t=0}^{2\pi/\omega} \left(-A\omega\sin(\omega t - \phi)\right) \times F_{ts}^{dissipative} dt$$

$$Virial = \int_{t=0}^{2\pi/\omega} x(t) \times F_{ts}(d,\dot{d})dt = \int_{t=0}^{2\pi/\omega} \left(A\cos(\omega t - \phi)\right) \times F_{ts}^{conservative} dt$$

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Origin of amplitude reduction

$$\frac{\ddot{x}}{\omega_0^2}+x+\frac{1}{\omega_0 Q}\dot{x}=\frac{1}{k}\left(F_0\cos(\omega t)+F_{ts}(d,\dot{d})\right)\ \textit{where}\quad \omega_0=\sqrt{\frac{k}{m}},\ Q=\frac{m\omega_0}{c}$$

$$x(t)=A\cos(\omega t-\phi)\ \textit{so that}\quad d(t)=Z+A\cos(\omega t-\phi)\ \textit{and}\ \dot{x}(t)=\dot{d}(t)=-A\omega\sin(\omega t-\phi)\quad (1)$$

$$-\left(\left(\frac{\omega}{\omega_0}\right)^2-1\right)\cos(\omega t-\phi)-\left(\frac{\omega}{\omega_0 Q}\right)\sin(\omega t-\phi)=\frac{1}{kA}\left\{F_0\cos(\omega t)+F_{ts}(d,\dot{d})\right\}\qquad (2)$$

$$\int_{t=0}^{2\pi/\omega}\sin(\omega t-\phi)\times(\square)\,dt\Rightarrow$$

$$\frac{1}{kA}F_0\sin(\phi)=\left(\frac{\omega}{\omega_0 Q}\right)+\frac{\omega}{\pi kA}\int_{t=0}^{2\pi/\omega}\sin(\omega t-\phi)\times F_{ts}(d,\dot{d})dt=\left(\frac{\omega}{\omega_0 Q_{eff}}\right)\qquad (3)$$

$$\textit{with}\ \frac{1}{Q_{eff}}=\frac{1}{Q}+\frac{\omega_0}{\pi kA}\int_{t=0}^{2\pi/\omega}\sin(\omega t-\phi)\times F_{ts}(d,\dot{d})dt$$

$$\int_{t=0}^{2\pi/\omega}\cos(\omega t-\phi)\times(\square)\,dt\Rightarrow$$

$$\frac{F_0\cos(\phi)}{kA}=\omega_{eff}^2-\left(\frac{\omega}{\omega_0}\right)^2\qquad (4)$$

$$\textit{with}\ \omega_{eff}^2=1-\frac{\omega}{\pi kA}\int_{t=0}^{2\pi/\omega}\cos(\omega t-\phi)\times F_{ts}(d,\dot{d})dt$$

Combining (3) (4)

$$A=\frac{F_0/k}{\sqrt{\left(\left(\frac{\omega}{\omega_0}\right)^2-\omega_{eff}{}^2\right)^2+\left(\frac{\omega}{\omega_0 Q_{eff}}\right)^2}}\qquad (5)$$

- *L. Wang et al*

App. Phys. Lett., ***73***, *3781, 1998*

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Origin of amplitude reduction

$$x(t) = A\cos(\omega t - \phi) \text{ so that}$$

$$d(t) = Z + A\cos(\omega t - \phi) \text{ and } \dot{x}(t) = \dot{d}(t) = -A\omega\sin(\omega t - \phi) \quad (1)$$

$$A = \frac{F_0 / k}{\sqrt{\left(\left(\frac{\omega}{\omega_0}\right)^2 - \omega_{eff}{}^2\right)^2 + \left(\frac{\omega}{\omega_0 Q_{eff}}\right)^2}}$$

$$\text{with } \frac{1}{Q_{eff}} = \frac{1}{Q} + \frac{\omega_0}{\pi k A}\int_{t=0}^{2\pi/\omega} \sin(\omega t - \phi) \times F_{ts}(d,\dot{d})dt = \frac{1}{Q} - \frac{\omega_0}{\omega \pi k A^2}E_{diss}$$

$$\text{and } \omega_{eff}^2 = 1 - \frac{\omega}{\pi k A}\int_{t=0}^{2\pi/\omega} \cos(\omega t - \phi) \times F_{ts}(d,\dot{d})dt$$

- Amplitude decreases due to both conservative and dissipative interactions
- Topography map in tapping mode is not a map of constant conservative or dissipative interaction

Average force

$$\frac{F_0\cos(\phi)}{kA} = \omega_{eff}^2 - \left(\frac{\omega}{\omega_0}\right)^2 \qquad (1)$$

with $\omega_{eff}^2 = 1 - \frac{\omega}{\pi k A}\int_{t=0}^{2\pi/\omega}\cos(\omega t - \phi)\times F_{ts}^{conservative}dt$

Consider

$$\omega_{eff}^2 = 1 - \frac{\omega}{\pi k A}\int_{t=0}^{2\pi/\omega}\cos(\omega t - \phi)\times F_{ts}(d)dt = 1 - \frac{\omega}{\pi k A^2}\int_{t=0}^{2\pi/\omega}A\cos(\omega t - \phi)\times F_{ts}(Z + A\cos(\omega t - \phi))dt$$

□ $1 + \frac{\omega}{\pi k A}\frac{2\pi}{\omega}\langle F_{ts}(d)\rangle = 1 + \frac{2}{kA}\langle F_{ts}(d)\rangle \qquad (2)$

Valid when contact time << oscillation period

$$\frac{2}{kA}\langle F_{ts}(d)\rangle = \frac{F_0\cos(\phi)}{kA} - 1 + \left(\frac{\omega}{\omega_0}\right)^2 \qquad (3)$$

*San Paulo and Garcia PRB, **64**, 2001*

When $\omega = \omega_0$

$$\langle F_{ts}(d)\rangle = \frac{F_0\cos(\phi)}{2} = \frac{F_0}{2}\sqrt{1-\sin^2\phi} = \frac{F_0}{2}\sqrt{1-\left(\frac{A}{A_0}\right)^2} = \frac{kA}{2Q}\sqrt{1-\left(\frac{A}{A_0}\right)^2} \qquad (4)$$

- Depends only on cantilever properties and operating conditions !!

Implications of average force result

- Contact time and max contact force change over the sample so as to keep mean force constant
- Average force can be measured directly in commercial AFM systems
- Can also be used to distinguish between attractive and repulsive regimes of oscillating in tapping mode AFM