Lecture 14
Introduction to dynamic AFM

Arvind Raman
Mechanical Engineering
Birck Nanotechnology Center
- Cantilever driven near resonance
- The cantilever's resonant frequency, phase and amplitude are affected by short-scale force gradients
- In Amplitude Modulated AFM (AM-AFM) or tapping mode, driving frequency is fixed while cantilever approaches the sample
- In Frequency Modulated AFM (FM-AFM) the phase and amplitude are held constant while approaching the sample
The point mass model

Continuous AFM cantilever

\[ w(x,t) = A \sin(\omega t) \psi(x) \]

\[ \theta/\delta = ? \]

\[ \psi(x) = \cos(\beta \frac{x}{L}) - \cosh(\beta \frac{x}{L}) - \frac{\cos(\beta) + \cosh(\beta)}{\sin(\beta) + \sinh(\beta)} \left[ \sin(\beta \frac{x}{L}) - \sinh(\beta \frac{x}{L}) \right] \]

Point mass model

\[ w(x,t) = A \sin(\omega t) \psi(x) \]

\[ \psi(x) = - \frac{L^3}{6EI} \left( \frac{x}{L} \right)^3 x^3 + \frac{1}{2} \frac{L^3}{EI} \left( \frac{x}{L} \right)^2 \]

- Tip is massive, cantilever inertia negligible
- Replace cantilever by a spring of spring constant = static bending stiffness of lever
- Cantilever oscillates such that \( \theta/\delta = 2L/3 \)
Point mass model - free oscillations

\[ m\ddot{x} = -kx - cx \quad \text{or} \quad \ddot{x} + \frac{1}{\omega_0^2}x + \frac{1}{\omega_0Q} \dot{x} = 0 \quad (1) \]

where \( \omega_0 = \sqrt{\frac{k}{m}} \), \( Q = \frac{m\omega_0}{c} = \frac{\sqrt{mk}}{c} \)

General solution of type

\[ x(t) = e^{it} \Rightarrow \lambda^2 + \frac{\lambda}{\omega_0Q} = 0 \Rightarrow \lambda_{1,2} = -\frac{\omega_0}{2Q} \pm \frac{\omega_0}{2Q} \sqrt{1 - \frac{1}{4Q^2}} \quad (2) \]

\[ x(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}, \quad \text{integration constants to be determined from} \quad x(0), \dot{x}(0) \]

\[ x(t) = e^{-\frac{\omega_0}{2Q}t} \left( x(0) \cos \left( \sqrt{1 - \frac{1}{4Q^2}} \omega_0 t \right) + \frac{\dot{x}(0) + \omega_0 x(0)}{2Q} \sin \left( \sqrt{1 - \frac{1}{4Q^2}} \omega_0 t \right) \right) \quad (3) \]

if \( Q > \frac{1}{2} \) Underdamped oscillation

\[ x(t) = c_1 e^{-\frac{\omega_0}{2Q}t + \omega_0 \sqrt{1 - \frac{1}{4Q^2}} t} + c_2 e^{-\frac{\omega_0}{2Q}t - \omega_0 \sqrt{1 - \frac{1}{4Q^2}} t} \quad \text{if} \quad Q < \frac{1}{2} \quad \text{Overdamped oscillation} \quad (4) \]

- Damped natural frequency is different from natural frequency
- \( Q \) can be regarded as number of oscillation cycles before transients become small
Forced vibrations

- Mechanical (acoustic or piezo excitation)
- Magnetic excitation
- Magnetostrictive excitation
- Photothermal excitation
- Lorentz force excitation
- Ultrasound excitation
- Direct piezoelectric excitation

a. Acoustic excitation

b. Magnetic excitation
Response of acoustically excited levers

\[ m \ddot{x} = -k(x - y) - c\dot{x} \]

\[ \frac{\ddot{x}}{\omega_0^2} + x + \frac{1}{\omega_0 Q} \dot{x} = y(t); \text{ with } \omega_0 = \sqrt{\frac{k}{m}}, \quad Q = \frac{m \omega_0}{c} \]

Measured motion \( z(t) = x(t) - y(t) \)

\[ \frac{\ddot{z}}{\omega_0^2} + z + \frac{1}{\omega_0 Q} \dot{z} = -\frac{\dot{y}}{\omega_0^2} - \frac{1}{\omega_0 Q} \dot{y} \]

\( y(t) = Y_0 \sin(\omega t) \)

\( z^p(t) = A \sin(\omega t + \phi_{\text{acoustic}}) \)

\[ |H_{\text{acoustic}}(\omega)| = \frac{A}{Y_0} = \left( \frac{r^4 + (r/Q)^2}{(1-r^2)^2 + (r/Q)^2} \right)^{1/2} \]

\[ \phi_{\text{acoustic}}(\omega) = \tan^{-1}\left( \frac{Q}{r(1 + Q^2 r^2 - Q^2)} \right) \]

where \( r = \frac{\omega}{\omega_0} \)

- \( \omega_0 \) is the natural freq, \( \omega \) is the drive freq
- Maximum amplitude occurs when \( \omega > \omega_0 \)
- Base motion amplitude \( Y_0(r=1) = A/Q \) when \( Q \) is large!
- For \( Q=100 \), see response above
- Asymmetric peak, amplitude greater when \( \omega > \omega_0 \)
Response of directly excited AFM levers

\[
m\ddot{x} = -kx - cx + F_{\text{mag}}(t)
\]

\[
\frac{\ddot{x}}{\omega_0^2} + x + \frac{1}{\omega_0 Q} \dot{x} = \frac{1}{k} F_{\text{mag}}(t); \text{ with } \omega_0 = \sqrt{\frac{k}{m}}, Q = \frac{m\omega_0}{c}
\]

Measured motion = \(x(t)\)

\[
F_{\text{mag}}(t) = F_0 \sin(\omega t)
\]

\[
x^p(t) = A \sin(\omega t + \phi_{\text{magnetic}})
\]

\[
|H_{\text{mag}}(\omega)| = \frac{A}{F_0 / k} = \left(\frac{1}{(1 - r^2)^2 + (r / Q)^2}\right)^{1/2}
\]

\[
\phi_{\text{mag}}(\omega) = \tan^{-1}\left(\frac{r}{Q(r^2 - 1)}\right)
\]

where \(r = \frac{\omega}{\omega_0}\)

- \(\omega_0\) is the natural freq, \(\omega\) is the drive freq
- Maximum amplitude occurs when \(\omega < \omega_0\)
- For \(\omega << \omega_0\) \(A = F_{\text{mag}} / k\)
Asymmetric response with greater amplitude when $\omega < \omega_0$!

Classical phase response
Driven point mass model with tip-sample interaction

Highly nonlinear ordinary differential equation

Magnetic

\[ m\ddot{x} = -kx - cx + F_{\text{mag}}(t) + F_{\text{ts}}(Z + x(t)) \]

\[ \ddot{x} + x + \frac{1}{\omega_0^2} \dot{x} = \frac{1}{k} \left( F_{\text{mag}}(t) + F_{\text{ts}}(Z + x(t)) \right) \]

with \( \omega_0 = \sqrt{\frac{k}{m}} \), \( Q = \frac{m\omega_0}{c} \)

Measured motion = \( x(t) \)

\( F_{\text{mag}}(t) = F_0 \sin(\omega t) \)

Acoustic excitation

\[ \ddot{z} + z + \frac{1}{\omega_0^2} \dot{z} = -\ddot{y} - \frac{1}{\omega_0^2} \dot{y} + \frac{F_{\text{ts}}(Z + y(t) + x(t))}{\omega_0^2} \]
Linearized analysis

\[
\frac{\ddot{x}}{\omega_0^2} + x + \frac{1}{\omega_0 Q} \dot{x} = \frac{1}{k} \left( F_0 \sin(\omega t) + F_{ts}(Z + x(t)) \right); \quad d(t) = Z + x(t) \quad (1)
\]

At a given \(Z\) the equilibrium deflection is

\[
x^* = \frac{1}{k} F_{ts}(Z + x^*) \quad \text{where} \quad d^* = Z + x^* \quad (2)
\]

Let \(x(t) = x^* + \bar{x}(t)\) \quad (3)

Include time-dependent terms

\[
\frac{(\dddot{x} + \dddot{x}^*)}{\omega_0^2} + (\ddot{x} + x^*) + \frac{1}{\omega_0 Q} (\dot{x} + \dot{x}^*) = \frac{1}{k} \left( F_0 \sin(\omega t) + F_{ts}(Z + x^* + \bar{x}) \right) \quad (4)
\]

\[
\frac{\dddot{x}}{\omega_0^2} + (\ddot{x} + x^*) + \frac{1}{\omega_0 Q} \dot{x} = \frac{1}{k} \left( F_0 \sin(\omega t) + F_{ts}(Z + x^* + \bar{x}) \right) \quad (5)
\]

If \(\bar{x} \ll Z + x^*\) or when \(\bar{x} \ll d^*\) then

\[
\frac{\dddot{x}}{\omega_0^2} + (\ddot{x} + x^*) + \frac{1}{\omega_0 Q} \dot{x} = \frac{1}{k} \left( F_0 \sin(\omega t) + F_{ts}(Z + x^*) + \frac{\partial F_{ts}(d)}{\partial d} \bigg|_{d=d^*} \bar{x} \right) \quad (6)
\]

\[
\Rightarrow \quad \frac{\dddot{x}}{\omega_0^2} + \left( 1 - \frac{1}{k} \frac{\partial F_{ts}(d)}{\partial d} \bigg|_{d=x^*} \right) \ddot{x} + \frac{1}{\omega_0 Q} \dot{x} = \frac{1}{k} \left( F_0 \sin(\omega t) \right) \quad (7)
\]
Linearized analysis

\[
\frac{\ddot{x}}{\omega_0^2} + \left(1 - \frac{1}{k} \frac{\partial F_{ts}(d)}{\partial d}\right)_{d=d^*} \frac{1}{\omega_0 Q} \dot{x} + \frac{\omega_0}{Q} \ddot{x} = \frac{1}{k} \left(F_0 \sin(\omega t)\right)
\]

Or

\[
\ddot{x} + \omega_0^2 \left(1 - \frac{1}{k} \frac{\partial F_{ts}(d)}{\partial d}\right)_{d=d^*} \frac{1}{\omega_0} \dot{x} = \omega_0 \frac{\omega_0^2}{k} \left(F_0 \sin(\omega t)\right)
\]

\[
\omega_0^2 = \omega_0^2 \left(1 - \frac{1}{k} \frac{\partial F_{ts}(d)}{\partial d}\right)_{d=d^*}
\]

- When \(\frac{\partial F_{ts}}{\partial d}\bigg|_{d=d^*} > 0\) attractive force and natural frequency decreases

- When \(\frac{\partial F_{ts}}{\partial d}\bigg|_{d=d^*} < 0\) rep. regime and natural frequency increases

\[
\dot{F}_{ts}(d) = k(x^* - d^*)
\]

\[
F = kx
\]
Limitations of the linearized analysis

For what driven oscillation amplitude is this approximation valid?
Experiments with conventional tips

- **Si tip / HOPG sample z=90 nm, frequency sweep**

![Diagram of experimental setup]

- When brought closer to the sample, the tip sometimes sticks to the sample

Next class

- Nonlinear response
- Dynamic Approach curves