# Lecture 17
# Cantilever eigenmodes, equivalent point mass oscillator, analytical approaches

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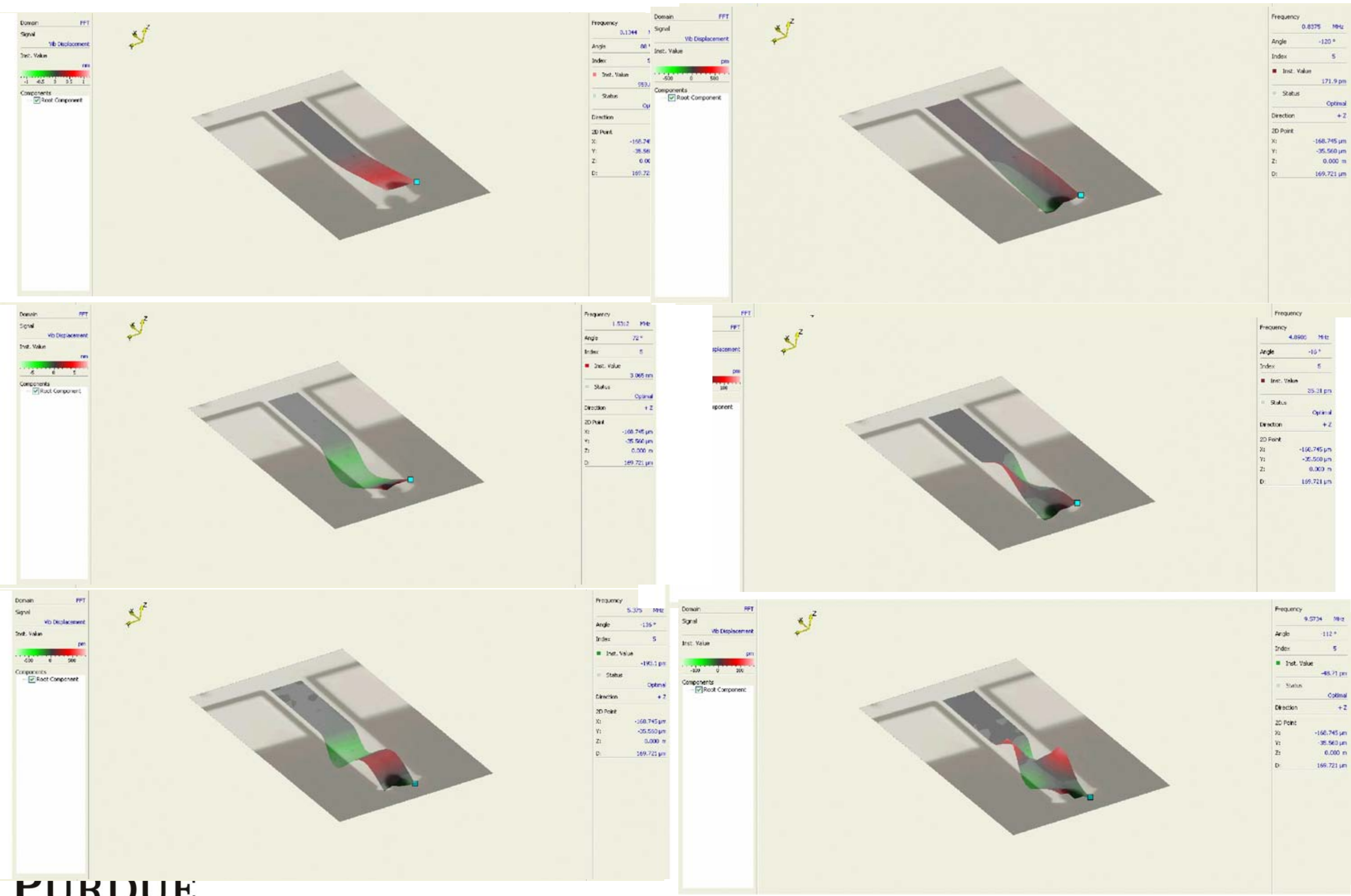
# Cantilever eigenmodes



# Point mass vs. continuous oscillator?

- The point mass model was derived with the assumption that cantilever mass was « tip mass
- The shape of the oscillating beam in the point mass model is assumed to be that of a statically bent beam under a tip force
- The point mass model does not predict any oscillation modes beyond the fundamental
- How to include spatially continuous nature of the AFM cantilever and yet enjoy the simplicity of a point mass model?

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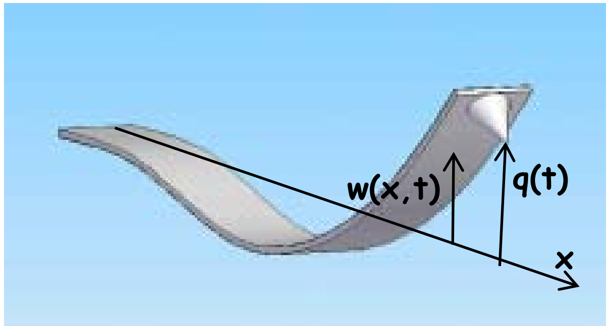
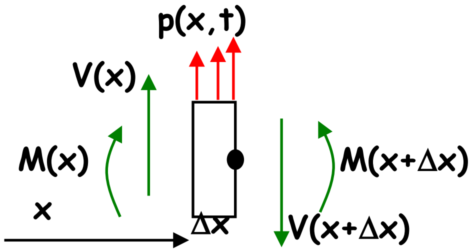
# Transverse vibrations of classical beam




p(x,t): external force per unit length

A: Area of cross section

ρ: mass density of cantilever

- Bernoulli-Euler beam theory

$$V(x) + p(x,t)\Delta x - V(x+\Delta x) = (\rho A \Delta x)\ddot{w},$$

$$\text{as } \Delta x \to 0 \text{ we get } \rho A \ddot{w} = -\frac{\partial V}{\partial x} + p(x,t)$$

*or*

$$\rho A \ddot{w} = -\frac{\partial^2 M}{\partial x^2} + p(x,t) = -EI\frac{\partial^4 w}{\partial x^4} + p(x,t)$$

*Or*

$$\rho A \ddot{w} + EI\frac{\partial^4 w}{\partial x^4} = p(x,t)$$

*To be solved with boundary conditions*

$$w(0) = 0$$

$$\frac{\partial w}{\partial x}(0) = 0$$

$$V(L) = EI\frac{\partial^3 w}{\partial x^3}(L) = ??$$

$$M(L) = EI\frac{\partial^2 w}{\partial x^2}(L) = ??$$

# Transverse vibrations of classical beam

- To calculate eigenmodes and natural frequencies, one can set p(x,t)=0 and any damping=0

$$\rho A\ddot{w} + EI\frac{\partial^4 w}{\partial x^4} = 0 \qquad (1)$$

$$Let \quad w(x,t) = \phi(x)T(t) \qquad (2)$$

$$\frac{\left(EI/\rho A\right)}{\phi(x)}\frac{d^4\phi(x)}{dx^4} = -\frac{1}{T(t)}\frac{d^2T}{dt^2} = const = \omega^2 \qquad (3)$$

$$T(t) = A\sin(\omega t) + B\cos(\omega t) \qquad \phi(x) = Ce^{\lambda x} \qquad (4)$$

$$(4)\ in\ (3a) \rightarrow$$

$$\lambda^4 = \beta^4 = \frac{\rho A\omega^2}{EI} \Rightarrow \lambda_{1,2} = \pm\beta,\ \lambda_{3,4} = \pm i\beta \qquad (5)$$

$$\phi(x) = C_1 e^{\beta x} + C_2 e^{-\beta x} + C_3 e^{i\beta x} + C_4 e^{-i\beta x}$$

$$\cong C_1\sin(\beta x) + C_2\cos(\beta x) + C_3\sinh(\beta x) + C_4\cosh(\beta x)$$

$$and \quad \omega = \beta^2\sqrt{\frac{EI}{\rho A}}$$

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# Transverse vibrations of classical beam

- Assuming negligible tip mass

$$\phi(x) = C_1\cos(\beta x) + C_2\sin(\beta x) + C_3\cosh(\beta x) + C_4\sinh(\beta x) \; where \; \beta^4 = \rho A\omega^2 / EI \quad (1)$$

$$w(0) = 0\,, \quad \frac{\partial w}{\partial x}(0) = 0, \quad EI\frac{\partial^3 w}{\partial x^3}(L) = 0, \quad EI\frac{\partial^2 w}{\partial x^2}(L) = 0$$

$$C_1 = C_3 = 0 \; and$$

$$C_2(\cos(\beta L) + \cosh(\beta L)) + C_4(\sin(\beta L) + \sinh(\beta L)) = 0 \quad (2)$$

$$C_2(-\sin(\beta L) + \sinh(\beta L)) + C_4(\cos(\beta L) + \cosh(\beta L)) = 0$$

$$or \begin{bmatrix} \cos(\beta L) + \cosh(\beta L) & \sin(\beta L) + \sinh(\beta L) \\ -\sin(\beta L) + \sinh(\beta L)) & \cos(\beta L) + \cosh(\beta L) \end{bmatrix} \begin{bmatrix} C_2 \\ C_4 \end{bmatrix} \quad (3)$$

$$for \; solutions \; where \; C_2, C_4 \neq 0 \; we \; must \; have$$

$$\cos(\beta L)\cosh(\beta L) + 1 = 0 \quad (4)$$

$$and \quad C_4 = -\frac{\cos(\beta L) + \cosh(\beta L)}{\sin(\beta L) + \sinh(\beta L)} C_2 \quad (5)$$

$$Solving \; (4) \; yields \; (\beta L)_1 = 1.875, (\beta L)_2 = 4.694, (\beta L)_3 = 7.855\ldots\ldots$$

*So that the eigen modes are*

$$\phi_n(x) = (\cos(\beta_n x) - \cosh(\beta_n x)) - \frac{\cos(\beta_n L) + \cosh(\beta_n L)}{\sin(\beta_n L) + \sinh(\beta_n L)}\left(\sin(\beta_n x) - \sinh(\beta_n x)\right) \quad (6)$$

$$Normalize \; so \; that \; \phi_n(L) = 1$$

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# Eigenmodes

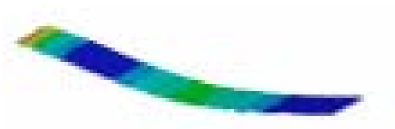
1st eigenmode

$$(\beta L)_1 = 1.875, (\beta L)_2 = 4.694, (\beta L)_3 = 7.855\ldots\ldots$$

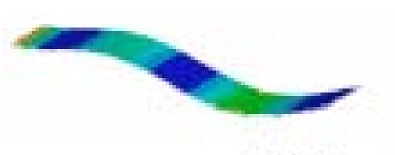
*Thus*

$$\omega_1 : \omega_2 : \omega_3 : .... = 1 : 6.26 : 17.55 : ...$$

*for a uniform rectangular lever with negligible tip mass*



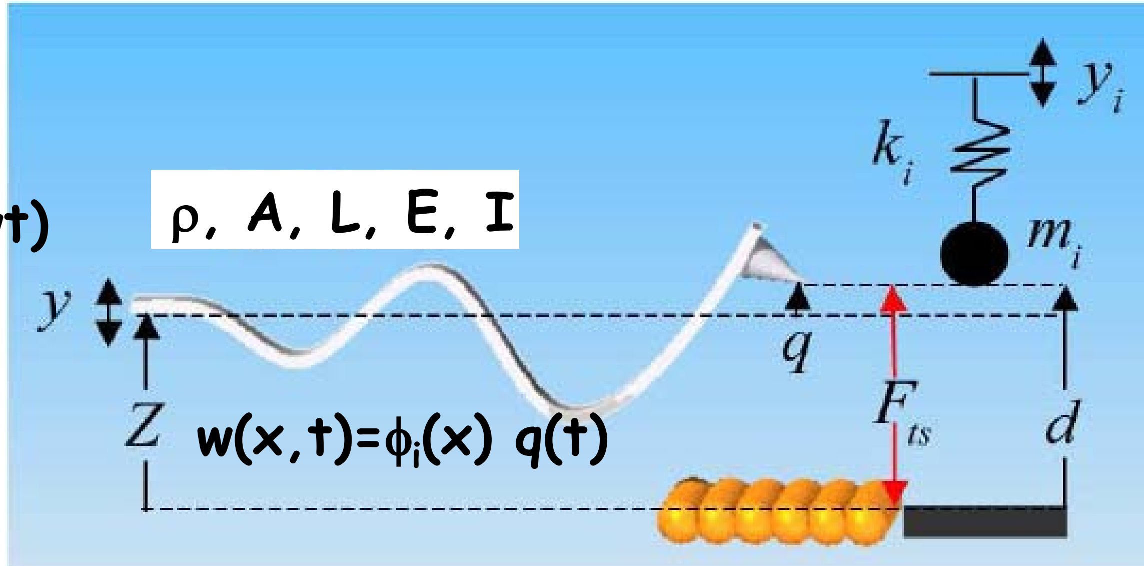
2nd eigenmode



3rd eigenmode

# Equivalent point mass oscillator

$y(t)=Y\cos(wt)$




$y_i(t)=Y_i\cos(wt)$

- Energy based equivalence principle

$$\frac{1}{2}k_i q^2 = \frac{1}{2}\int_{x=0}^{x=L} EI\left(\frac{d^2\phi_i}{dx^2}\right)^2 dx \quad \frac{1}{2}m_i q^2 = \frac{1}{2}\int_{x=0}^{x=L} \rho A\left(\phi_i\right)^2 dx \quad Y_i = Y\left(\frac{\omega}{\omega_i}\right)^2 \frac{\int_0^L \phi_i(x)dx}{\int_0^L \left(d^2\phi_i \big/ dx^2\right) dx}$$

*For negligible tip mass*

$$k_1 = 1.03k,\ k_2 = 40.5k,\ k_3 = 317k$$

$$m_1 = m_2 = m_3 = .... = 0.249\ \rho AL$$

$$Y_1 \sim 1.5Y$$

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Melcher *et al*, App. Phys. Lett. 91(5), 2007

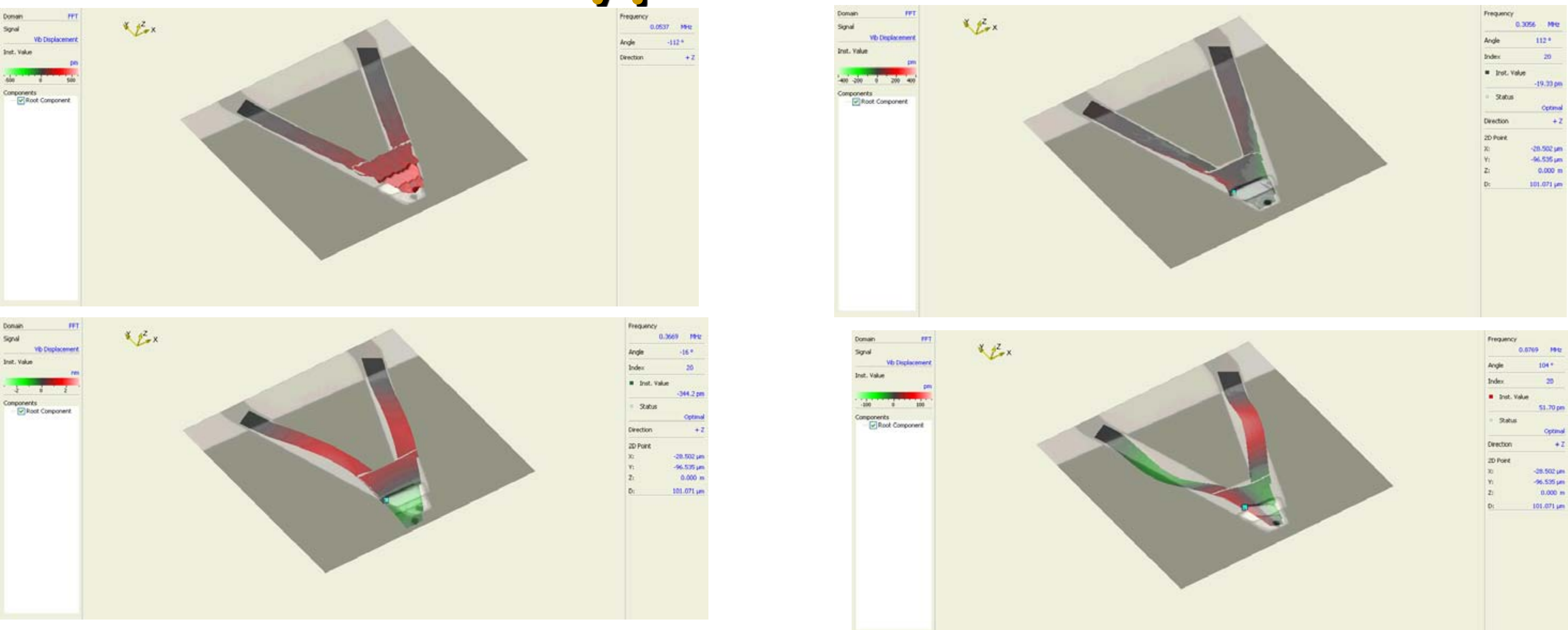
# Effect of tip mass

- Not much effect on first mode shape
- Big effect on second and higher mode shape
- Often $k_2$~60-80k etc.

R. Tung, T. Wutscher, D. Martinez-Martin, R. Reifenberger, F. Giessibl, A. Raman,
*Journal of Applied Physics*, 107(10), 104508, 2010.

D. Kiracofe, A. Raman,
*Journal of Applied Physics*, 108, 034320, 2010

# Other types of cantilevers



- Vibrations of triangular cantilevers can be thought of as two cantilevers joined together at their tip
- Leads to symmetric and anti-symmetric eigenmodes
- However the point mass oscillator equivalence holds

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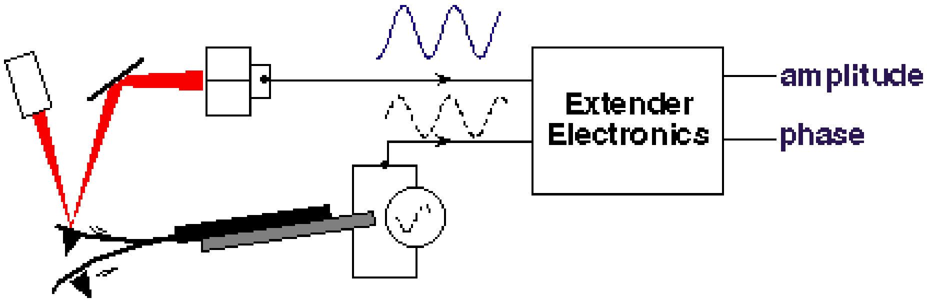
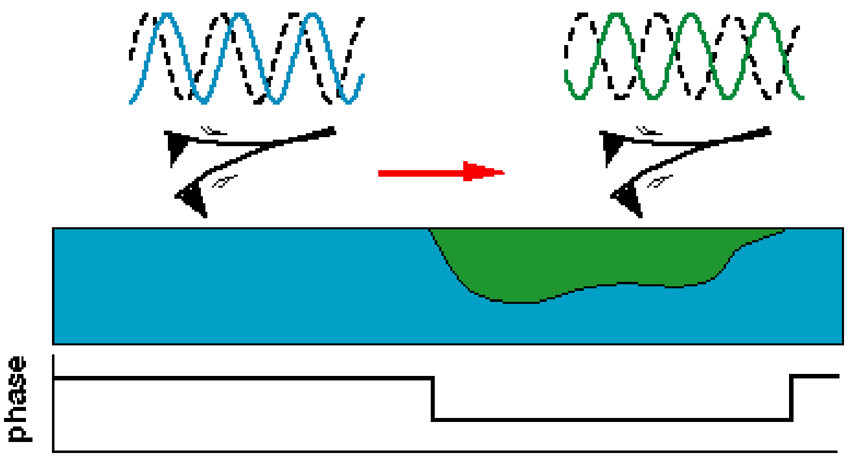
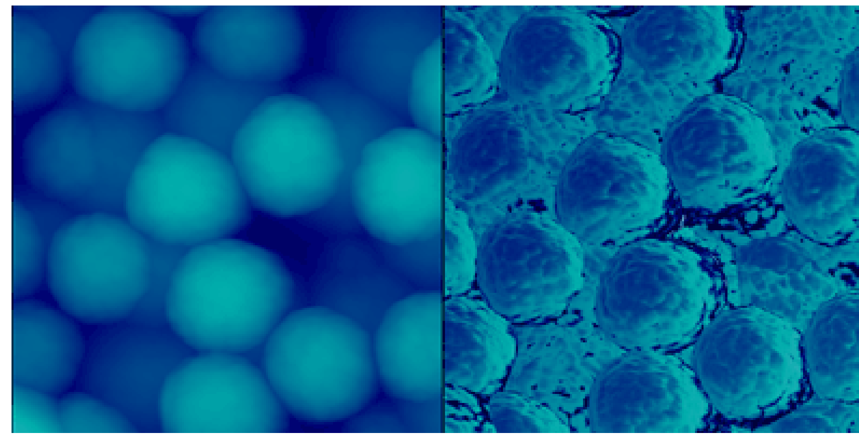
# Analytical descriptions of AM-AFM

- So far we have resorted to numerical simulations (VEDA) of the point mass model or linearized the equations
- Perturbation methods are quite useful too to help understand
  - Origin of phase contrast
  - Origin of amplitude reduction
  - Average forces while tapping

# Phase Contrast



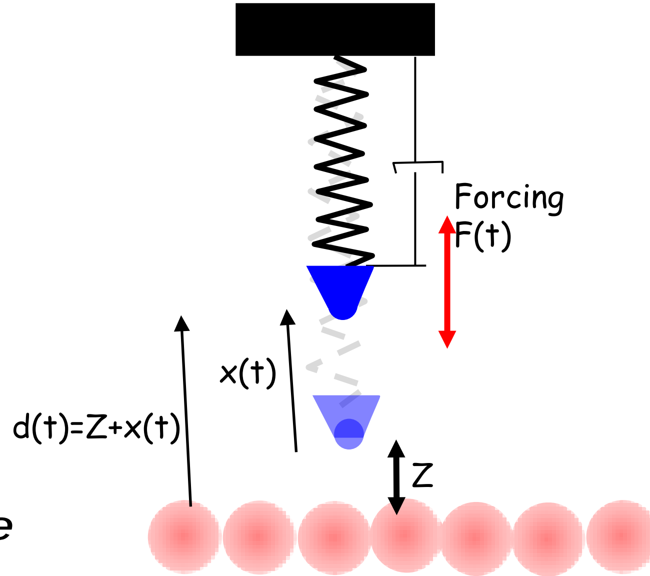



AFM height (left) and phase (right) images of poly(methylmethacrylate)

(Veeco, Inc.)

- Regular tapping mode implemented but signal phase monitored
- But what does a phase contrast image mean really?

# Analytical description of AM-AFM




$$m\ddot{x} = -kx - c\dot{x} + F_0 \cos(\omega t) + F_{ts}(d,\dot{d})$$

$$\frac{\ddot{x}}{\omega_0^2} + x + \frac{1}{\omega_0 Q}\dot{x} = \frac{1}{k}\left(F_0 \cos(\omega t) + F_{ts}(d,\dot{d})\right) \quad where \qquad (1)$$

$$with \quad \omega_0 = \sqrt{\frac{k}{m}}, Q = \frac{m\omega_0}{c}$$

$$Let \quad x(t) = A\cos(\omega t - \phi) \quad so\ that \quad \dot{x}(t) = \dot{d}(t) = -A\omega \sin(\omega t - \phi) \qquad (2)$$

$\phi$ *is phase lag*    *A is the setpoint amplitude*

*Substitute* (2) *in* (1), *we get*

$$-\left(\left(\frac{\omega}{\omega_0}\right)^2 - 1\right)\cos(\omega t - \phi) - \left(\frac{\omega}{\omega_0 Q}\right)\sin(\omega t - \phi) = \frac{1}{kA}\left\{F_0 \cos(\omega t) + F_{ts}(d,\dot{d})\right\} \qquad (3)$$

# Energy dissipation

$$x(t) = A\cos(\omega t - \phi) \quad \textit{so that} \quad \dot{x}(t) = \dot{d}(t) = -A\omega\sin(\omega t - \phi) \qquad (1)$$

$$-\left(\left(\frac{\omega}{\omega_0}\right)^2 - 1\right)\cos(\omega t - \phi) - \left(\frac{\omega}{\omega_0 Q}\right)\sin(\omega t - \phi) = \frac{1}{kA}\left\{F_0\cos(\omega t) + F_{ts}(d,\dot{d})\right\} \qquad (2)$$

$$\int_{t=0}^{2\pi/\omega}\sin(\omega t - \phi)\times(\square)dt \Rightarrow -\left(\frac{\omega}{\omega_0 Q}\right)\frac{\pi}{\omega} = -\frac{1}{kA}\frac{\pi}{\omega}F_0\sin(\phi) + \frac{1}{kA}\int_{t=0}^{2\pi/\omega}\sin(\omega t - \phi)\times F_{ts}(d,\dot{d})dt$$

$$Or,\ \sin(\phi) = \frac{kA}{F_0}\left\{\frac{\omega}{Q\omega_0} - \frac{1}{\pi kA^2}\int_{t=0}^{2\pi/\omega}\left(-A\omega\sin(\omega t - \phi)\right)\times F_{ts}(d,\dot{d})dt\right\} = \frac{kA}{F_0}\left\{\frac{\omega}{Q\omega_0} - \frac{1}{\pi kA^2}E_{diss}\right\}$$

(3)

$$But\ \frac{kA_0}{F_0} = \sqrt{\left(1 - \left(\frac{\omega}{\omega_0}\right)^2\right)^2 + \left(\frac{\omega}{\omega_0 Q}\right)^2} \quad \textit{so we get}$$

$$\sin(\phi) = \frac{1}{\sqrt{\left(1 - \left(\frac{\omega}{\omega_0}\right)^2\right)^2 + \left(\frac{\omega}{\omega_0 Q}\right)^2}}\left\{\frac{\omega}{Q\omega_0}\frac{A}{A_0} - \frac{1}{\pi kAA_0}E_{diss}\right\} \qquad (4)$$

$$If\ \omega = \omega_0\ then\ \sin(\phi) = \left\{\frac{A}{A_0} - \frac{Q}{\pi kAA_0}E_{diss}\right\}$$

- $A/A_0$= constant in tapping mode scan
- Sin(ϕ) contrast = energy dissipation contrast!

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*Anczykowski et al App. Surf. Sci.,* ***140****, 376, 1999*

# Analytical description of AM-AFM

- Conversely

$$E_{diss} = \pi k A A_0 \left\{ \frac{\omega}{Q\omega_0}\frac{A}{A_0} - \sin(\phi)\sqrt{\left(1-\left(\frac{\omega}{\omega_0}\right)^2\right)^2 + \left(\frac{\omega}{\omega_0 Q}\right)^2} \right\} \quad (1)$$

*If* $\omega = \omega_0$

$$E_{diss} = \frac{\pi k}{Q}\left\{A^2 - A_0 A \sin(\phi)\right\} \quad (2)$$

- Eq. (7) relates the energy dissipated per cycle due to tip-sample interaction to observables
- By quantitative knowledge of Q, k, A, $A_0$, and $\phi$ it becomes p ossible to know in an experiment the energy dissipated in eV or pJ per cycle

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# The virial

$$x(t) = A\cos(\omega t - \phi) \quad \text{so that} \quad \dot{x}(t) = \dot{d}(t) = -A\omega \sin(\omega t - \phi) \qquad (1)$$

$$-\left(\left(\frac{\omega}{\omega_0}\right)^2 - 1\right)\cos(\omega t - \phi) - \left(\frac{\omega}{\omega_0 Q}\right)\sin(\omega t - \phi) = \frac{1}{kA}\left\{F_0 \cos(\omega t) + F_{ts}(d,\dot{d})\right\} \qquad (2)$$

$$\int_{t=0}^{2\pi/\omega} \cos(\omega t - \phi)\times(\square)dt \Rightarrow -\left(\left(\frac{\omega}{\omega_0}\right)^2 - 1\right)\frac{\pi}{\omega} = \frac{1}{kA}\frac{\pi}{\omega}F_0\cos(\phi) + \frac{1}{kA}\int_{t=0}^{2\pi/\omega}\cos(\omega t - \phi)\times F_{ts}(d,\dot{d}$$

$$Or, \cos(\phi) = \frac{kA}{F_0}\left\{-\left(\left(\frac{\omega}{\omega_0}\right)^2 - 1\right) - \frac{\omega}{\pi k A^2}\int_{t=0}^{2\pi/\omega}\left(A\cos(\omega t - \phi)\right)\times F_{ts}(d,\dot{d})dt\right\}$$

$$Or, \cos(\phi) = \frac{kA}{F_0}\left\{-\left(\left(\frac{\omega}{\omega_0}\right)^2 - 1\right) - \frac{2}{kA^2}\langle F_{ts} \square x\rangle\right\}$$

virial

(3)

$$\cos(\phi) = \frac{1}{\sqrt{\left(1 - \left(\frac{\omega}{\omega_0}\right)^2\right)^2 + \left(\frac{\omega}{\omega_0 Q}\right)^2}}\left\{-\left(\left(\frac{\omega}{\omega_0}\right)^2 - 1\right) - \frac{2}{kA^2}\langle F_{ts} \square x\rangle\right\} \qquad (4)$$

$$If \ \omega = \omega_0 \ then \ \cos(\phi) = -\frac{2Q}{kA^2}\langle F_{ts} \square x\rangle$$

- Cos(ϕ) maps virial (a measure of conservative interactions)

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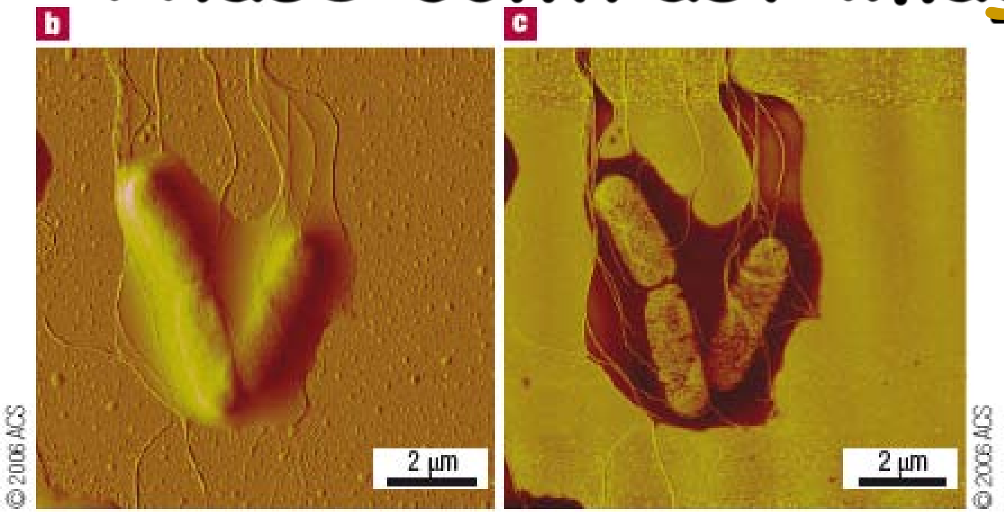
# Last comments on phase

- Pay attention to whether your system plots phase lead or phase lag (How?)
- Always make sure to plot phase in degrees not in Volts
- Only when you plot sin($\phi$) is it a map of dissipation, with energy dissipation scaling as sin($\phi$)-$A/A_0$
- When cos($\phi$) is plotted, it is a map of local conser vative interactions
- In a sense the oscillator settles to a value of phase lag which is a balance between conservative and dissipative interactions
- In liquids even the interpretation of sin($\phi$) is different

# Phase contrast imaging in air/vacuum




($\phi$). **b**,**c**, The above considerations are illustrated by comparing the topography an aggregate of three *Salmonella typhimurium* cells covered by an extracellula polymeric capsule (**b**) and the phase image (**c**), that is acquired simultaneously the topography, reveals the inner structure of the cell as well as the continuity flagellae. (R. Avci *et al.* ref. 49 © 2007 American Chemical Society).

- *From Garcia et al*

*Nature Materials, 6, 2007*


**Figure 4** Complex microdomain structure of a block copolymer. The AFM images are rendered into three-dimensions using the height image as height-field and the phase image as contrast. The images show the formation of terraces in a thin film of SBS block copolymer and the systematic change of microdomain structures along the changes in film thickness from 32 nm at the lowest terrace to 57 nm at the higher terrace. Reprinted with permission from ref. 54.

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# Next time

- Please Read Garcia and Perez in the reader
- Other analytical results (average force, peak force, amplitude reduction)