(c) Relaxation time approximation (RTA)

- Let us write the non-equilibrium distribution function as:

$$f(\vec{r},\vec{k},t) = \underbrace{f_S(\vec{r},\vec{k},t)}_{\text{symmetric term in the momentum (large)}} + \underbrace{f_A(\vec{r},\vec{k},t)}_{\text{asymmetric term in momentum (small term)}}$$

For non-degenerate sc, the collision integral is of the form:

$$\left.\frac{\partial f}{\partial t}\right|_{coll} = \sum_{\vec{k}'} \left[S(\vec{k}',\vec{k}) f(\vec{k}') - S(\vec{k},\vec{k}') f(\vec{k}) \right] =$$

$$= \overbrace{\sum_{\vec{k}'} \left[S(\vec{k}',\vec{k}) f_S(\vec{k}') - S(\vec{k},\vec{k}') f_S(\vec{k}) \right]}^{(\partial f_S/\partial t)|_{coll}} + \underbrace{\sum_{\vec{k}'} \left[S(\vec{k}',\vec{k}) f_A(\vec{k}') - S(\vec{k},\vec{k}') f_A(\vec{k}) \right]}_{\left.\frac{\partial f_A}{\partial t}\right|_{coll}}$$

- We now consider two cases:

  (a) Equilibrium conditions: $f_S = f_0$; $f_A = 0 \Rightarrow \left.\frac{\partial f}{\partial t}\right|_{coll} = \left.\frac{\partial f_S}{\partial t}\right|_{coll} = 0$

  (b) Non-equilibrium conditions: $f_A \neq 0$. In this case, we must consider two different conditions:

  (1) Low-field condition $\rightarrow$ f_S retains its equilibrium form with $T_c = T_L$. In this case: $\left.\frac{\partial f_S}{\partial t}\right|_{coll} = 0$

  (2) High-field conditions $\rightarrow$ $T_c \neq T_L$ and f_S does not retain its equilibrium form. In this case: $\left.\frac{\partial f_S}{\partial t}\right|_{coll} \neq 0$.

In both case (a) and case (b), a plausible form for the term $(\partial f_A/\partial t)|_{coll}$ is:

$$\left.\frac{\partial f_A}{\partial t}\right|_{coll} = -\frac{f_A}{\tau_f}$$

where τ_f is a characteristic time that describes how the distribution function relaxes to its equilibrium form.

- With the discussion presented in the previous page, we may conclude that:
  - → at low-fields: $\left.\frac{\partial f}{\partial t}\right|_{coll} = \left.\frac{\partial f_A}{\partial t}\right|_{coll} = -\frac{f_A}{\tau_f}$
  - → at high-fields: $\left.\frac{\partial f}{\partial t}\right|_{coll} = \left.\frac{\partial f_S}{\partial t}\right|_{coll} + \left.\frac{\partial f_A}{\partial t}\right|_{coll} = \left.\frac{\partial f_S}{\partial t}\right|_{coll} - \frac{f_A}{\tau_f}$

- To understand the meaning of the relaxation time, we consider a semiconductor in which there are no spatial and momentum gradients. for this case, the BTE becomes:

$$\frac{\partial f}{\partial t} = \left.\frac{\partial f_A}{\partial t}\right|_{scatt} = -\frac{f_A}{\tau_f} = -\frac{f-f_0}{\tau_f} \quad \text{(at low fields)}$$

i.e.

$$\frac{\partial f}{\partial t} + \frac{f}{\tau_f} = \frac{f_0}{\tau_f}$$

The solution of this first-order differential equation is:

$$f(t) = f_0 + [f(0) - f_0]\, e^{-t/\tau_f}$$

This result suggests that any perturbation in the system will decay exponentially with a characteristic time constant τ_f. It also suggests that the RTA is only good when $[f(0) - f_0]$ is not very large.

Note: Important restriction for the relaxation-time approximation to be valid is that τ_f is independent of the distribution function and the applied electric field.

(D) Solving the BTE in the RTA

- Let us consider a SC that is uniformly-doped (no spatial gradients). In this case $\nabla_r f = 0$. Under steady-state conditions, we also have $\partial f/\partial t = 0$. With the above simplifications, the BTE reduces to:

$$\vec{F}\cdot\nabla_p f = \frac{1}{\hbar}\vec{F}\cdot\nabla_k f = \left.\frac{\partial f}{\partial t}\right|_{coll}$$

- For parabolic bands and spatial gradients along one direction, one can expand the distribution function into Legendre polynomials:

$$f(z,p) = f_0(z,p) + \sum_{n=1}^{\infty} g_n(E)\,P_n(\cos\theta)$$

where : $P_0 = 1$, $P_1 = \cos\theta$, $P_2 = \frac{3}{2}\cos^2\theta - \frac{1}{2}$, ... In the above expression, θ is the angle between the applied electric field (along the symmetry axis), and the momentum of carriers. For low fields, and no spatial gradients, we have:

$$f(p) \cong f_0(p) + \underbrace{g_1(p)\cos\theta}_{f_A(p)}$$

Substituting this result on the LHS of the BTE gives:

$$LHS = -\frac{e}{\hbar}\vec{\mathcal{E}}\cdot\nabla_k\left[f_0(p) + g_1(p)\cos\theta\right] \approx -\frac{e}{\hbar}\vec{\mathcal{E}}\cdot\nabla_k f_0(p)$$

For parabolic dispersion relation and isotropic mass:

$$\nabla_k f_0 = \frac{\partial f_0}{\partial E}\nabla_k E = \frac{\partial f_0}{\partial E}\frac{\hbar^2}{m^*}\vec{k} = \frac{\partial f_0}{\partial E}\hbar\vec{v}$$

Therefore, the LHS can be written as:

$$LHS \cong -\frac{e}{\hbar}\vec{\mathcal{E}}\cdot\vec{v}\,\hbar\,\frac{\partial f_0}{\partial E} = -e\,(\vec{\mathcal{E}}\cdot\vec{v})\,\frac{\partial f_0}{\partial E} = -e\,\mathcal{E}\,v\cos\theta\left(\frac{\partial f_0}{\partial E}\right),$$

where, as previously noted θ is the angle between the electric field and $\vec{v}$.

- We now consider the collision integral on the RHS:

$$\frac{\partial f}{\partial t}\bigg|_{coll} = \sum_{\vec{k}'} \underbrace{\left[S(\vec{k}',\vec{k}) f_0(\vec{k}_1) - S(\vec{k},\vec{k}') f_0(\vec{k}) \right]}_{\text{This term vanishes by detailed balance.}}$$

$$+ \sum_{\vec{k}'} \left[S(\vec{k}',\vec{k}) g_1(k') \cos\theta' - S(\vec{k},\vec{k}') g_1(k) \cos\theta \right]$$

$$= \sum_{\vec{k}'} S(\vec{k},\vec{k}') g_1(k) \cos\theta \left[\frac{S(\vec{k}',\vec{k}) g_1(\vec{k}') \cos\theta'}{S(\vec{k},\vec{k}') g_1(\vec{k}) \cos\theta} - 1 \right]$$

$$= - g_1(k) \cos\theta \sum_{\vec{k}'} S(\vec{k},\vec{k}') \left[1 - \frac{S(\vec{k}',\vec{k}) g_1(\vec{k}')}{S(\vec{k},\vec{k}') g_1(\vec{k})} \frac{\cos\theta'}{\cos\theta} \right]$$

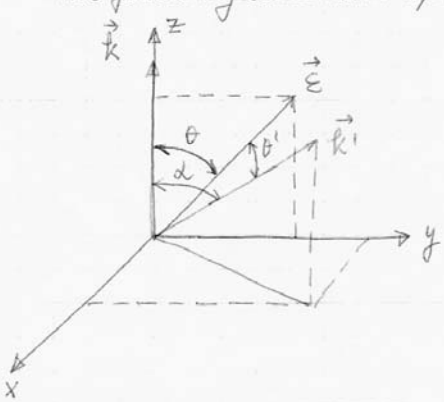
The use of the principle of detailed balance gives:

$$\frac{S(\vec{k}',\vec{k})}{S(\vec{k},\vec{k}')} = \frac{f_0(k)}{f_0(k')}$$

or:

$$\frac{\partial f}{\partial t}\bigg|_{coll} = - g_1(k) \cos\theta \sum_{\vec{k}'} S(\vec{k},\vec{k}') \left[1 - \frac{f_0(k) g_1(k')}{f_0(k') g_1(k)} \frac{\cos\theta'}{\cos\theta} \right]$$

We can further simplify the above result, by considering the following coordinate system:



$$\vec{k} = (0, 0, k)$$
$$\vec{k}' = (k' \sin\alpha \cos\varphi,\ k' \sin\alpha \sin\varphi,\ k' \cos\alpha)$$
$$\vec{\varepsilon} = (0,\ \varepsilon \sin\theta,\ \varepsilon \cos\theta)$$

Then:

$$\vec{\varepsilon} \cdot \vec{k}' = \varepsilon k' \cos\theta'$$
$$= \varepsilon k' (\underbrace{\sin\alpha \sin\varphi \sin\theta}_{\text{integration over } \varphi \text{ will make this term to vanish}} + \cos\alpha \cos\theta)$$

Hence: $\dfrac{\cos\theta'}{\cos\theta} = tg\,\theta \sin\alpha \sin\varphi + \cos\alpha \rightarrow \cos\alpha$

To summarize:

$$\left.\frac{\partial f}{\partial t}\right|_{coll} = -g_1(k)\cos\theta \sum_{\vec{k}'} S(\vec{k},\vec{k}') \underbrace{\left[1 - \frac{f_0(k)g_1(k')}{f_0(k')g_1(k)}\cos\alpha\right]}$$

For the RTA approximation to be valid, this term should lead to a relaxation time that does not depend upon the distribution function.

- Consider now the case of ELASTIC scattering process. Then:

$$|\vec{k}| = |\vec{k}'| \quad \text{and} \quad \frac{f_0(k)g_1(k')}{f_0(k')g_1(k)} = 1, \text{ which gives:}$$

$$\left.\frac{\partial f}{\partial t}\right|_{coll} = -g_1(k)\cos\theta \sum_{\vec{k}'} S(\vec{k},\vec{k}')(1-\cos\alpha) = -\frac{g_1(k)\cos\theta}{\tau_m(k)} = -\frac{f_A}{\tau_m(k)}$$

Hence, when the scattering process is ELASTIC, the characteristic time τ_f equals the momentum relaxation time.

- If the scattering process is ISOTROPIC, then $S(\vec{k},\vec{k}')$ does not depend upon α. In this case, the second term in the square brackets averages to zero and

$$\left.\frac{\partial f}{\partial t}\right|_{coll} = -g_1(k)\cos\theta \sum_{\vec{k}'} S(\vec{k},\vec{k}') = -\frac{g_1(k)\cos\theta}{\tau(k)} = -\frac{f_A}{\tau(k)}$$

Thus, in this case, characteristic time is the scattering time, or the average time between collision events.

- To summarize, under low-field conditions, when the scattering process is either ISOTROPIC or ELASTIC, the collision term can be represented as $-f_A/\tau_f(k)$, where in general $\tau_f(k)$ is the momentum relaxation time that depends only upon the nature of the scattering process.

- The BTE can, thus, be written as:

$$-e\mathcal{E}v\cancel{\cos\theta}\left(\frac{\partial f_0}{\partial E}\right) = -\frac{g_1(k)\cancel{\cos\theta}}{\tau_m(k)}$$

or:

$$g_1(k) = e\mathcal{E}v\,\tau_m(k)\left(\frac{\partial f_0}{\partial E}\right)$$

The distribution function is, thus, equal to:

$$f(k) = f_0(k) + e\mathcal{E}v\cos\theta\,\tau_m(k)\left(\frac{\partial f_0}{\partial E}\right)$$

Assuming that $\vec{\mathcal{E}} = \mathcal{E}\vec{i_z}$, $v\cos\theta = v_z$ and we have:

$$f(k) = f_0(k) + e\mathcal{E}v_z\,\tau_m(k)\,\frac{\partial f_0}{\partial E}$$

- We now want to investigate the form of this distribution function. For this purpose, we use:

$$\frac{\partial f_0}{\partial E} = \frac{\partial f_0}{\partial k_z}\frac{\partial k_z}{\partial E} = \frac{\partial f_0}{\partial k_z}\frac{1}{\hbar^2 k_z/m^*} = \frac{1}{\hbar v_z}\frac{\partial f_0}{\partial k_z}$$

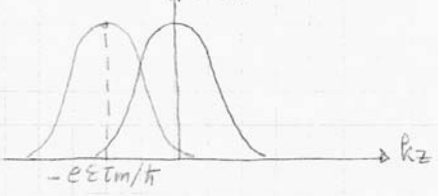
which leads to:

$$f(k) = f_0(k) + e\mathcal{E}\cancel{v_z}\,\tau_m(k)\frac{1}{\hbar\cancel{v_z}}\frac{\partial f_0}{\partial k_z} = f_0(k) + \underbrace{\frac{e}{\hbar}\mathcal{E}\tau_m(k)\frac{\partial f_0}{\partial k_z}}$$

Looks like the linear term in Taylor-series expansion of $f(k)$

Hence:

$$f(k_x, k_y, k_z) = f_0\left(k_x, k_y, k_z + \frac{e}{\hbar}\mathcal{E}\tau_m(k)\right]$$



Note that the assumption made in arriving at this result is that $\mathcal{E}$ is small. Hence, displaced Maxwellian is a good representation of the D.F. under low-field conditions.