This document provides a matlab script that calculates eigenvalues for Cartesian geometries developed by A. Haji-Sheikh and J.V. Beck (X_IJ, I,J=1,2,3). Also provided are verification examples and data. The program is in Table 1.

Table 2 contains a program to test the subroutine. It gives the eigenvalues and a check of the values provided the eigenvalues are neither zero nor greater than 1.0e+14.

Table 3 gives computed eigenvalues for Bi_1 = 1 and Bi_1 = 2. Notice that the column called "check" gives zero values as it should.

Table 4 gives computed eigenvalues for Bi_1 = 0 and Bi_1 = 2. Notice that the column called "check" gives zero values as it should. See also Abramowitz and Stegun, p. 225.

Table 5 gives computed eigenvalues for Bi_1 = 1.0e+15 and Bi_1 = 2. Notice that the column called "check" gives zero values as it should. See also Abramowitz and Stegun, p. 224.

Jim Beck, March 2, 2009

Table 1. Matlab subroutine for the computation of the eigenvalues for the XIJ cases with I and J = 1, 2 or 3. It is based on the method developed by A. Haji-Sheikh and J. Beck

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%feigXIJ.m 6/20/07 Matlab subroutine written by James V. Beck based on programming and method of Prof. A. Haji-Sheikh,
  %Haji-Sheikh, A. and Beck, J.V., "An Efficient Method of Computing Eigenvalues in Heat Conduction",
  %Numerical Heat Transfer Part B: Fundamentals, Vol  38 No. 2, pp. 133-156, 2000.
%It gives the eigenvalues for the X11, X12, X13, ..., X33 cases. X33 is the general case and includes the others.
%However, the X11, X12, X21, X22 eigenvalues are well-known, for X11 and X22, the eigenvalues are n *pi, for n = 1, 2, 3, ... .
%(Also n = 0 for X22.)  For X12 and X21, the eigenvalues are (2*n-1)*pi/2 for n = 1, 2, 3,
function XIJ=feigXIJa(m,B1,B2)
%Input quantities
%B1 is the one of the Biot numbers. For X31, it is for Bi_1.
%B2 is the other Biot number. For X13, it is for Bi_2. Actually the eigenvalues for X13 and X31 are the same.
  %For the boundary condition of the first kind (given temperature), Bi value goes to infinity. Use BI1=1.0e+15 or BI2 = 1.0e+15.
  %For the case of bc of the seoond kind, the Bi value goes to zero. Use BI1=0 or BI2 = 0
  %m is the number of eigenvalues
%Accuracy: Machine accuracy is expected, about 14 decimal place accuracy
BI1=B1;
BI2=B2;
AN=1:1:m;
%Immediately below generates the X33 eigenvalues using the Haji/Beck ubroutine
CN =(AN-0.75)*pi;
    DN=(AN-0.5)*pi;
    D=1.22*(AN-1)+0.76;
    P23 = 1./AN;
    BX = BI1*BI2;
    BS = BI1 + BI2;
    GAM = 1.0- 1.04*(sqrt((BS + BX+CN-pi/4.0)./(BS +BX+D))-(BS+BX+sqrt(D.*(CN-pi/4.0)))./(BS+BX+D));
    X23 = (BI1+BI2-CN)./(BI1+BI2+CN);
    X13 = BX./(BX+0.2+BS*(pi*pi*(AN-0.5)/2.0));
    G13=1+X13.*(1-X13).*(1-(0.85./AN)-(0.6-0.71./AN).*(X13+1).*(X13-0.6-0.25./AN));
    E23 = GAM.*(P23.*X23 + (1.0 - P23).*tanh(X23)/tanh(1)) ;
    E13 = 2.0*G13.*X13;
    Y23=CN+pi*E23/4.0;
    Y13=DN+pi*E13/4.0;
    ZN =(Y13*BX./(BX+DN.^2)+Y23.*(1.0-BX./(BX+DN.^2)));

    for i=1:3
    A0=((6.0+3.0*BS+BX-ZN.*ZN).*cos(ZN)-(6.0+BS)  *ZN.*sin(ZN))/6.0;
    B0=(ZN.^2-BS-BX).*cos(ZN)+(2.0+BS)*ZN.*sin(ZN);
    C0=(ZN.^2-BX).*sin(ZN)-BS*ZN.*cos(ZN);
    A0=(1.0+BS)*sin(ZN)+2.0*ZN.*cos(ZN)-C0/2.0;
    E0=-C0./B0-(-A0.*C0.^3+A0.*C0.^2 .*B0)./(3.0*A0.*C0.*C0.*B0  -2.0*A0.*C0.*B0.^2+B0.^4);
    ZN = ZN+E0;
    end

    if BI1 == 0 & BI2 == 0
      ZN(1)=0;
      end
    num=1:1:m;
    XIJ=[num' ZN'];
```

Table 2. Matlab program to test the feigXIJ subroutine

```
%EigenXIJtest.m 3 1 09 written by James V Beck
clear all
m=30;
B1=1.0e+15;
B2=1.0e+15;
B1=1;
B2=2;

X2=feigXIJ(m,B1,B2);

if B1 > 0 & B1 < 1.0e+14 &  B2 > 0 & B2 < 1.0e+14
   check=tan(X2(:,2))-X2(:,2)*(B1+B2)./(X2(:,2).^2-B1*B2);
end

%check=(X2(:,2).^2-B1*B2).*tan(X2(:,2))-X2(:,2)*(B1+B2);

sizeX2=size(X2);
B1v=B1*ones(sizeX2(1),1);
B2v=B2*ones(sizeX2(1),1);
if B1 > 0 & B1 < 1.0e+14 &  B2 > 0 & B2 < 1.0e+14
BB=[ B1v B2v X2(:,2) check   X2(:,2)/pi];
sprintf(' Bi_1         Bi_2     Eigenvalue     check    Eigenvalue/pi')
fprintf('%10.4e %10.4e %16.10f %16.10f   %16.10f \n',BB')
else
BB=[ B1v B2v X2(:,2)  X2(:,2)/pi];
sprintf(' Bi_1         Bi_2         Eigenvalue         Eigenv/pi')
fprintf('%10.4e %10.4e %16.10f %16.10f    \n',BB')

end
```

Table 3. Test results for $Bi_1 = 1$ and $Bi_1 = 2$.

Bi_1	Bi_2	Eigenvalue	check	Eigenvalue/pi
1.0000e+000	2.0000e+000	1.5094103447	-0.0000000000	0.4804602350
1.0000e+000	2.0000e+000	3.8712443675	-0.0000000000	1.2322553540
1.0000e+000	2.0000e+000	6.7201711094	-0.0000000000	2.1390969010
1.0000e+000	2.0000e+000	9.7299219095	0.0000000000	3.0971303356
1.0000e+000	2.0000e+000	12.7993457557	0.0000000000	4.0741582907
1.0000e+000	2.0000e+000	15.8959501170	0.0000000000	5.0598380725
1.0000e+000	2.0000e+000	19.0069585908	0.0000000000	6.0501028257
1.0000e+000	2.0000e+000	22.1264571633	-0.0000000000	7.0430700613
1.0000e+000	2.0000e+000	25.2513610230	0.0000000000	8.0377578532
1.0000e+000	2.0000e+000	28.3799115676	0.0000000000	9.0336064210
1.0000e+000	2.0000e+000	31.5110356081	0.0000000000	10.0302741580
1.0000e+000	2.0000e+000	34.6440421476	0.0000000000	11.0275411130
1.0000e+000	2.0000e+000	37.7784666053	-0.0000000000	12.0252594053
1.0000e+000	2.0000e+000	40.9139853116	-0.0000000000	13.0233260079
1.0000e+000	2.0000e+000	44.0503659538	0.0000000000	14.0216669731
1.0000e+000	2.0000e+000	47.1874375228	-0.0000000000	15.0202278672
1.0000e+000	2.0000e+000	50.3250713740	0.0000000000	16.0189677413
1.0000e+000	2.0000e+000	53.4631688911	0.0000000000	17.0178552047
1.0000e+000	2.0000e+000	56.6016532177	0.0000000000	18.0168657935
1.0000e+000	2.0000e+000	59.7404635761	-0.0000000000	19.0159801615
1.0000e+000	2.0000e+000	62.8795512758	0.0000000000	20.0151828099
1.0000e+000	2.0000e+000	66.0188768532	-0.0000000000	21.0144611771
1.0000e+000	2.0000e+000	69.1584079871	0.0000000000	22.0138049750
1.0000e+000	2.0000e+000	72.2981179535	-0.0000000000	23.0132056971
1.0000e+000	2.0000e+000	75.4379844654	-0.0000000000	24.0126562491
1.0000e+000	2.0000e+000	78.5779887892	-0.0000000000	25.0121506680
1.0000e+000	2.0000e+000	81.7181150639	0.0000000000	26.0116839052
1.0000e+000	2.0000e+000	84.8583497715	-0.0000000000	27.0112516575
1.0000e+000	2.0000e+000	87.9986813193	0.0000000000	28.0108502351
1.0000e+000	2.0000e+000	91.1390997098	-0.0000000000	29.0104764555

Table 4. Results for $Bi_1 = 0$ and $Bi_1 = 2$. See also Abramowitz and Stegun, p. 225.

Bi_1	Bi_2	Eigenvalue	check	Eigenvalue/pi
0.0000e+000	2.0000e+000	1.0768739863	-0.0000000000	0.3427796360
0.0000e+000	2.0000e+000	3.6435971674	0.0000000000	1.1597929997
0.0000e+000	2.0000e+000	6.5783337327	-0.0000000000	2.0939486617
0.0000e+000	2.0000e+000	9.6295603433	-0.0000000000	3.0651842569
0.0000e+000	2.0000e+000	12.7222987718	0.0000000000	4.0496334740
0.0000e+000	2.0000e+000	15.8336114149	-0.0000000000	5.0399950474
0.0000e+000	2.0000e+000	18.9546817665	-0.0000000000	6.0334625958
0.0000e+000	2.0000e+000	22.0814757673	0.0000000000	7.0287520383
0.0000e+000	2.0000e+000	25.2119030642	-0.0000000000	8.0251979948
0.0000e+000	2.0000e+000	28.3447768698	0.0000000000	9.0224226993
0.0000e+000	2.0000e+000	31.4793749203	0.0000000000	10.0201962480
0.0000e+000	2.0000e+000	34.6152330552	-0.0000000000	11.0183708940
0.0000e+000	2.0000e+000	37.7520396346	0.0000000000	12.0168474393
0.0000e+000	2.0000e+000	40.8895777660	-0.0000000000	13.0155568448
0.0000e+000	2.0000e+000	44.0276918992	-0.0000000000	14.0144495974
0.0000e+000	2.0000e+000	47.1662676028	0.0000000000	15.0134892724
0.0000e+000	2.0000e+000	50.3052188363	-0.0000000000	16.0126484822
0.0000e+000	2.0000e+000	53.4444796698	0.0000000000	17.0119062408
0.0000e+000	2.0000e+000	56.5839987379	-0.0000000000	18.0112461981
0.0000e+000	2.0000e+000	59.7237354324	0.0000000000	19.0106554280
0.0000e+000	2.0000e+000	62.8636572287	0.0000000000	20.0101235776
0.0000e+000	2.0000e+000	66.0037377708	0.0000000000	21.0096422575
0.0000e+000	2.0000e+000	69.1439554765	-0.0000000000	22.0092045980
0.0000e+000	2.0000e+000	72.2842925037	0.0000000000	23.0088049197
0.0000e+000	2.0000e+000	75.4247339745	0.0000000000	24.0084384869
0.0000e+000	2.0000e+000	78.5652673846	-0.0000000000	25.0081013192
0.0000e+000	2.0000e+000	81.7058821480	-0.0000000000	26.0077900471
0.0000e+000	2.0000e+000	84.8465692433	0.0000000000	27.0075017989
0.0000e+000	2.0000e+000	87.9873209347	0.0000000000	28.0072341123
0.0000e+000	2.0000e+000	91.1281305511	-0.0000000000	29.0069848639

Table 5. Results for Bi_1 = 1.0e+15 and Bi_1 = 2. See also Abramowitz and Stegun, p. 224.

Bi_1	Bi_2	Eigenvalue	check	Eigenvalue/pi
1.0000e+015	2.0000e+000	2.2889297281	0.0000000000	0.7285889612
1.0000e+015	2.0000e+000	5.0869850941	0.0000000000	1.6192376463
1.0000e+015	2.0000e+000	8.0961636032	0.0000000000	2.5770889151
1.0000e+015	2.0000e+000	11.1727058683	0.0000000000	3.5563827333
1.0000e+015	2.0000e+000	14.2763529183	0.0000000000	4.5443042726
1.0000e+015	2.0000e+000	17.3932439646	0.0000000000	5.5364415067
1.0000e+015	2.0000e+000	20.5175229099	0.0000000000	6.5309303822
1.0000e+015	2.0000e+000	23.6463238196	0.0000000000	7.5268586437
1.0000e+015	2.0000e+000	26.7780870756	0.0000000000	8.5237298492
1.0000e+015	2.0000e+000	29.9118938696	0.0000000000	9.5212515332
1.0000e+015	2.0000e+000	33.0471686947	0.0000000000	10.5192405059
1.0000e+015	2.0000e+000	36.1835330908	0.0000000000	11.5175762998
1.0000e+015	2.0000e+000	39.3207281323	0.0000000000	12.5161764964
1.0000e+015	2.0000e+000	42.4585707717	0.0000000000	13.5149828299
1.0000e+015	2.0000e+000	45.5969279841	0.0000000000	14.5139529569
1.0000e+015	2.0000e+000	48.7357007949	0.0000000000	15.5130553731
1.0000e+015	2.0000e+000	51.8748140534	0.0000000000	16.5122661572
1.0000e+015	2.0000e+000	55.0142096788	0.0000000000	17.5115668214
1.0000e+015	2.0000e+000	58.1538420786	0.0000000000	18.5109428532
1.0000e+015	2.0000e+000	61.2936749662	0.0000000000	19.5103827023
1.0000e+015	2.0000e+000	64.4336791037	0.0000000000	20.5098770619
1.0000e+015	2.0000e+000	67.5738306709	0.0000000000	21.5094183498
1.0000e+015	2.0000e+000	70.7141100665	0.0000000000	22.5090003269
1.0000e+015	2.0000e+000	73.8545010149	0.0000000000	23.5086178122
1.0000e+015	2.0000e+000	76.9949898892	0.0000000000	24.5082664683
1.0000e+015	2.0000e+000	80.1355651941	0.0000000000	25.5079426362
1.0000e+015	2.0000e+000	83.2762171650	0.0000000000	26.5076432076
1.0000e+015	2.0000e+000	86.4169374541	0.0000000000	27.5073655254
1.0000e+015	2.0000e+000	89.5577188827	0.0000000000	28.5071073044
1.0000e+015	2.0000e+000	92.6985552434	0.0000000000	29.5068665689

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