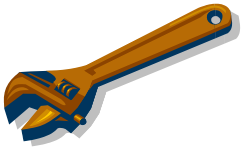
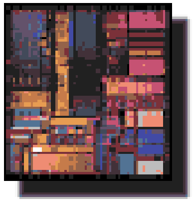
    

ECE 595Z
Digital Systems Design Automation

Module 2 (Lectures 3-5) : Advanced Boolean Algebra
Lecture 5



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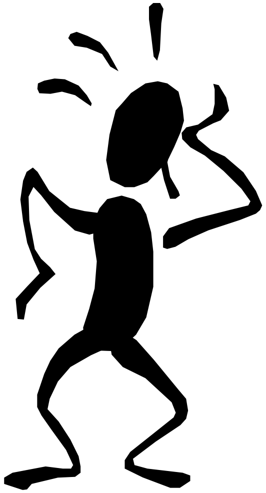
Terminology Checklist

- Boolean Algebra
- Boolean Function
- Cube
- Implicant (of a function)
- Minterm
- Cover (of a function)
- Tautology
- Satisfiable / Un-satisfiable
- Sum-of-products
- Minterm canonical representation
- Product-of-sums
- Conjunctive Normal Form
- Disjunctive Normal Form
- Binary Decision Tree
- Binary Decision Diagram
- Symbolic Simulation



Lecture #5 Outline

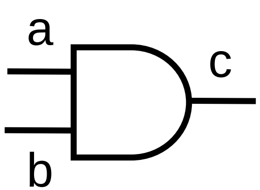
- Converting between representations - continued
- Co-factors and their applications

Conversion : Example #2

- How do you convert a general Boolean network (multi-level circuit) into a Boolean formula that is linear in the circuit size?
  - Size(formula) = O(M) where M = no. of gates in circuit
  - SOP may be exponential in the worst case
  - Hints
    - Use variables to represent intermediate signals in the circuit
    - Compose the formula using a 1 : 1 mapping from each gate in the circuit into a piece of the formula

Converting a Boolean Circuit into a CNF Formula

- First, let us see how very simple circuits (single gates) can be expressed in CNF form



b

a	∧	0	1
	0	0	0
	1	0	1

$c = ab$

⇓

$c \rightarrow ab,\ ab \rightarrow c$

⇓

$c \rightarrow a,\ c \rightarrow b,\ ab \rightarrow c$

$a = 0 \rightarrow c = 0$

$b = 0 \rightarrow c = 0$

$a = 1, b = 1 \rightarrow c = 1$

⇓

$a' \rightarrow c',\ b' \rightarrow c',\ ab \rightarrow c$

⇓

$(a+c')(b+c')(a'+b'+c)$

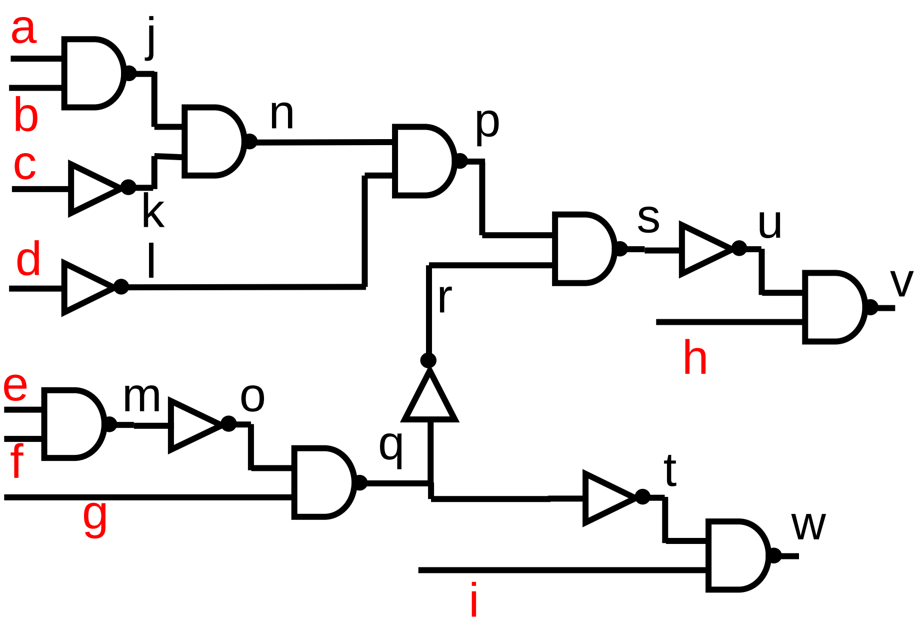
Converting a Boolean Circuit into a CNF Formula

- Simple rules for converting various basic gates into CNF equivalent

Gate Type	Function	CNF Formula
NOT	c = a’	(a+c)(a’+c’)
AND	c=ab	(a+c’)(b+c’)(a’+b’+c)
NAND	c=a’+b’	(a+c)(b+c)(a’+b’+c’)
OR	c=a+b	(a’+c)(b’+c)(a+b+c’)
NOR	c = a’b’	(a’+c’)(b’+c’)(a+b+c)

Converting a Boolean Circuit into a CNF Formula

- Now, we are ready to convert a multi-level circuit into a CNF formula
  - Simply concatenate formulae representing each of its gates



Known as the Tseitin Transformation

(a+j)(b+j)(a’+b’+j’)
(c+k)(c’+k’)
(d+l)(d’+l’)
(e+m)(f+m)(e’+f’+m’)
(m+o)(m’+o’)
(j+n)(k+n)(j’+k’+n’)
(n+p)(l+p)(n’+l’+p’)
(o+q)(g+q)(o’+g’+q’)
(q+r)(q’+r’)
(p+s)(r+s)(p’+r’+s’)
(s+u)(s’+u’)
(u+v)(h+v)(u’+h’+v’)
(q+t)(q’+t’)
(t+w)(i+w)(t’+l’+w’)

Co-factors

- A very useful operation on Boolean functions
- Applications of co-factors
  - Shannon's expansion
  - Boolean difference
  - Universal and Existential Quantification

Co-factors of Boolean Functions

- A co-factor of a function is derived by fixing one of the variables to a constant (0 or 1), resulting in a new function of n-1 variables
- Given a function $f(x_1 \ldots x_n)$
  - Positive co-factor w.r.t. x_i is defined as

    $f_{x_i}(x_1 \ldots x_{i-1}, x_{i+1} \ldots x_n) = f(x_1 \ldots x_{i-1}, \mathbf{x_i = 1}, x_{i+1} \ldots x_n)$
  - Negative co-factor w.r.t. xi is defined as

    $f_{x_i'}(x_1 \ldots x_{i-1}, x_{i+1} \ldots x_n) = f(x_1 \ldots x_{i-1}, \mathbf{x_i = 0}, x_{i+1} \ldots x_n)$

Examples:

$f = ab + bc + ac$

$f_a = 1.b + bc + 1.c = b + c$

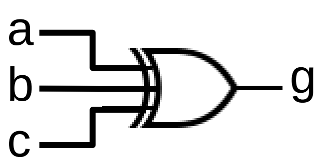
$f_{a'} =$

$f_b =$

$f_{b'} =$

$f_c =$

$f_{c'} =$



$g_a =$

$g_{a'} =$

$g_b =$

$g_{b'} =$

$g_c =$

$g_{c'} =$

Co-factors of Boolean Functions

- Also called
  - Shannon co-factors
  - Restriction of a function on a variable
- Can be applied on multiple variables

  $f_{x_i x_j'} = f(x_1 \ldots \mathbf{x_i = 1} \ldots \mathbf{x_j = 0} \ldots x_n)$
- Order does not matter

  $f_{x_i x_j} = (f_{x_i})_{x_j} = (f_{x_j})_{x_i}$
- Co-factor w.r.t. a cube

Examples:

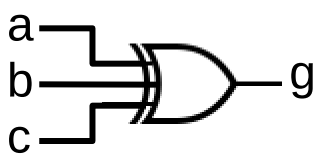
$f = ab + bc + ac$

$f_{ab} =$

$f_{ab'} =$

$f_{a'b'c'} =$

$f_{ab'c} =$



$g_{ab} =$

$g_{a'b} =$

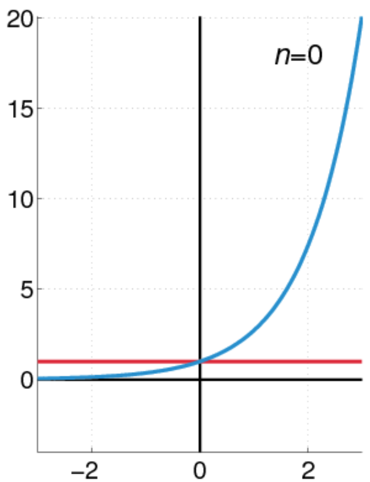
$g_{b'c'} =$

$g_{abc'} =$

OK, so why do we need Co-factors?

- Many applications… for example
- Recall **Taylor series** from high-school math?
  - A representation of a (real or complex) function as a sum of polynomial terms (1, x, x^2, x^3, x^4, …)
    - $e^x = 1 + x + x^2/2! + x^3/3! + \ldots$
  - General form:

    f =



Animation of Taylor series for e^x (Source: Wikipedia)

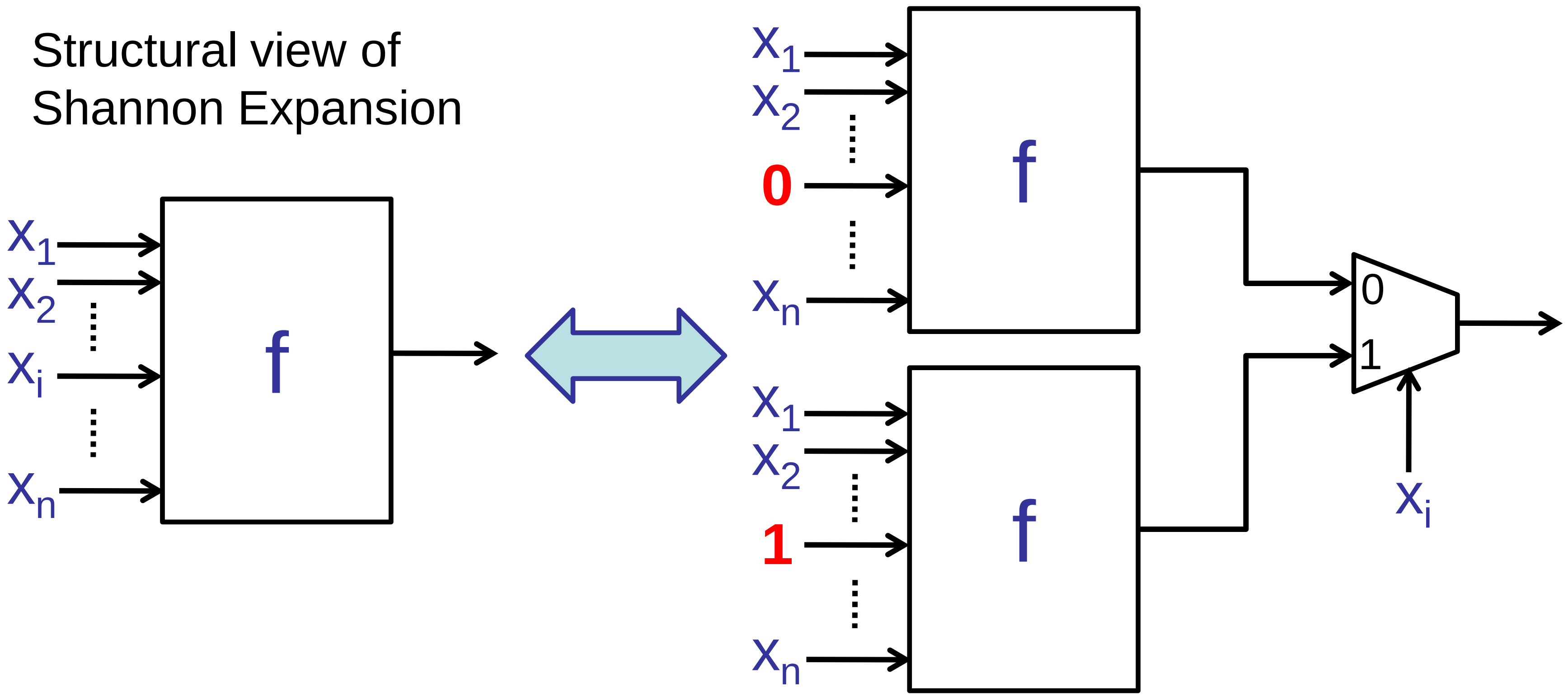
- Question: Is there a similar concept for Boolean functions?

Shannon's Expansion Theorem

- Given a Boolean function $f(x_1 \ldots x_n)$ and any variable x_i

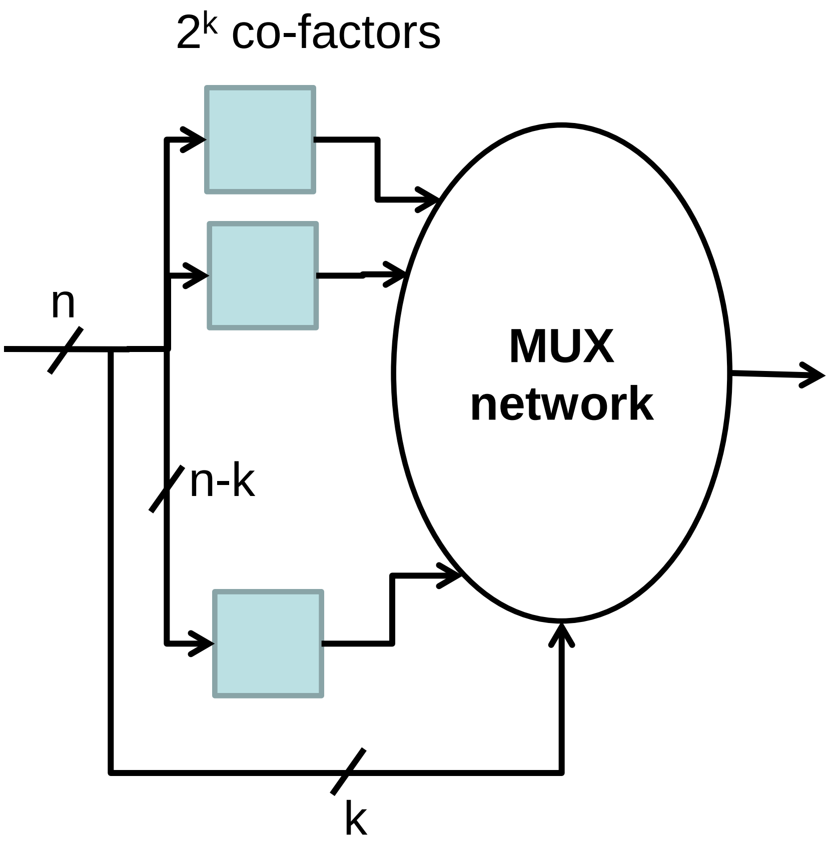
$$f = x_i f_{x_i} + x_{i'} f_{x_i'}$$

Structural view of Shannon Expansion



Shannon Expansion

- Also called **Shannon Decomposition**
- Can be applied recursively to "decompose" a function into it's co-factors
  - In the extreme case, just a network of multiplexers



For an interesting application to variation-tolerant synthesis, see:
Swaroop Ghosh, Swarup Bhunia and Kaushik Roy, "CRISTA: A new paradigm for low-power and robust circuit synthesis under parameter variations using critical path isolation", IEEE Trans. Computer Aided Design, Nov 2007.

Shannon Expansion

- Example

  $f = xy + zw' + x'w'$

Properties of Co-factors

- Given two functions f(x) and g(x)
- How can we compute co-factors of a function h that is derived from f and g?

Function	Co-factors
$h(x) = f'(x)$	$h_{x_i} = (f_{x_i})'$ $h_{x_i'} = (f_{x_i'})'$
$h(x) = f(x)$ AND $g(x)$	$h_{x_i} = f_{x_i}$ AND g_{x_i} $h_{x_i'} = f_{x_i'}$ AND $g_{x_i'}$
$h(x) = f(x)$ OR $g(x)$	$h_{x_i} = f_{x_i}$ OR g_{x_i} $h_{x_i'} = f_{x_i'}$ OR $g_{x_i'}$
$h(x) = f(x)$ XOR $g(x)$	$h_{x_i} = f_{x_i}$ XOR g_{x_i} $h_{x_i'} = f_{x_i'}$ XOR $g_{x_i'}$

Co-factor of complement is complement of co-factor

Co-factor of AND is AND of co-factors

Co-factor of OR is OR of co-factors

Co-factor of XOR is XOR of co-factors

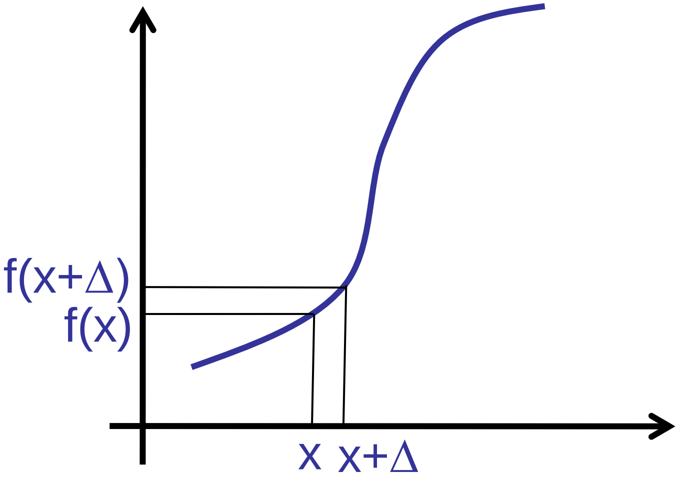
The co-factor operation distributes over any binary operator

Combinations of Co-factors

- Combining f_x and $f_{x'}$ in different ways leads to useful new functions
  - $f_x \oplus f_{x'}$ = ?
  - $f_x \cdot f_{x'}$ = ?
  - $f_x + f_{x'}$ = ?

Another analogy to the "real" world

- The derivative of a function measures how much it changes when it's input changes



$$f'(x) = \lim_{\Delta \to 0} \frac{f(x+\Delta) - f(x)}{\Delta}$$

- Let us think of the analogy in the case of Boolean functions (which only take values 0 and 1)

Boolean Difference

$$\frac{\partial f}{\partial x} = f_x \oplus f_{x'}$$

- Boolean difference of a function w.r.t. a variable is the exclusive-OR of the Shannon co-factors w.r.t. the variable
- Interpretation: $\frac{\partial f}{\partial x} = 1 \rightarrow$ f is sensitive to the value of x
- A new function that does not depend on x

Example:
f = xy + zw' + x'w'
f_x =
$f_{x'}$ =

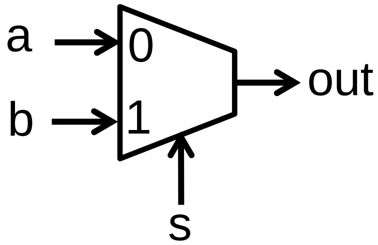
$$\frac{\partial f}{\partial x} =$$

Boolean Difference

- Examples:

$$\frac{\partial s}{\partial a} =$$

$$\frac{\partial c_{out}}{\partial c_{in}} =$$

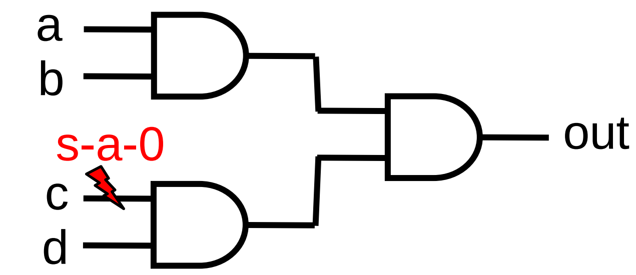


$$\frac{\partial out}{\partial a} =$$

$$\frac{\partial out}{\partial s} =$$

Application of Boolean Difference

- Manufacturing test
  - Apply test vectors to ensure that each fabricated instance of an IC behaves correctly
  - Cannot apply exhaustive test set (too big!)
- Fault model : Abstraction of physical defects that could impact the IC
  - Commonly used: "stuck-at" fault model
  - Signals in the circuit are stuck-at-0, stuck-at-1



How do you derive a test vector to detect the fault c s-a-0?

(i) Set c = 1
(ii) Set other inputs such that output of good and faulty circuits are different

Looks familiar?

Co-factors: Re-cap

- A very useful operation on Boolean functions
  - Derived by fixing one of the variables to a constant (0 or 1)
- Applications of co-factors
  - Shannon's expansion – a way to recursively simplify or divide Boolean functions
  - Boolean difference ($f_x \oplus f_{x'}$)
  - Universal and Existential Quantification

Quantification

- Two more functions of Shannon co-factors
  - $f_{x_i} \cdot f_{x_i'} = 1$ specifies when f = 1 independent of the value of x_i

    $f(x_1 \ldots x_{i-1}, \mathbf{x_i = 1}, x_{i+1} \ldots x_n) = 1$ AND

    $f(x_1 \ldots x_{i-1}, \mathbf{x_i = 0}, x_{i+1} \ldots x_n) = 1$
  - Called **Universal quantification** or **Consensus**

$$\forall x(f) = f_x \cdot f_{x'}$$

$C_a(f)$