DLL Algorithm

- Davis, Logemann and Loveland
- Basic framework for many modern SAT solvers
- Also known as DPLL for historical reasons

Basic DLL Procedure - DFS

(a' + b + c)
(a + c + d)
(a + c + d')
(a + c' + d)
(a + c' + d')
(b' + c' + d)
(a' + b + c')
(a' + b' + c)

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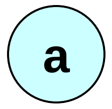

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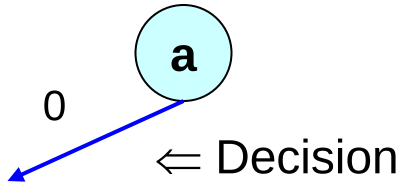

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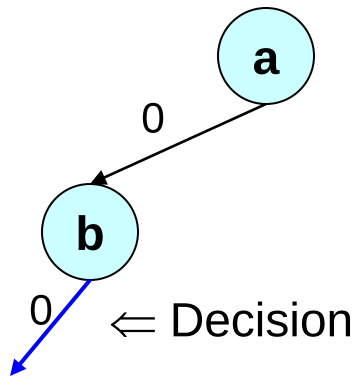

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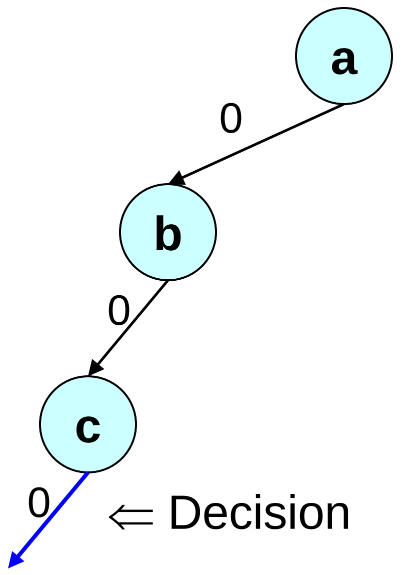

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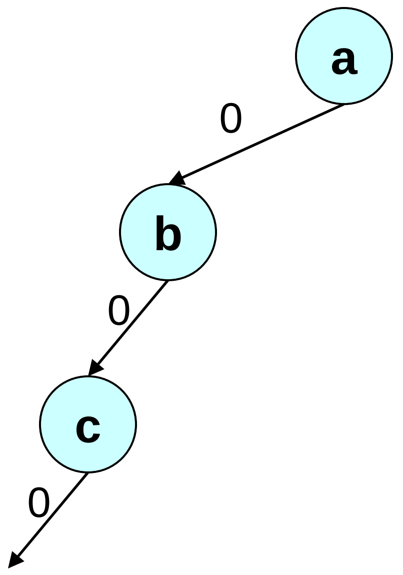

Implication Graph

$$\begin{tikzcd}
a=0 \arrow[r, "(a + c + d)"] \arrow[dr] & d=1 \arrow[d, leftrightarrow] \\
c=0 \arrow[r, "(a + c + d')"'] \arrow[ur] & d=0
\end{tikzcd}$$

Conflict!

Basic DLL Procedure - DFS

(a’ + b + c)
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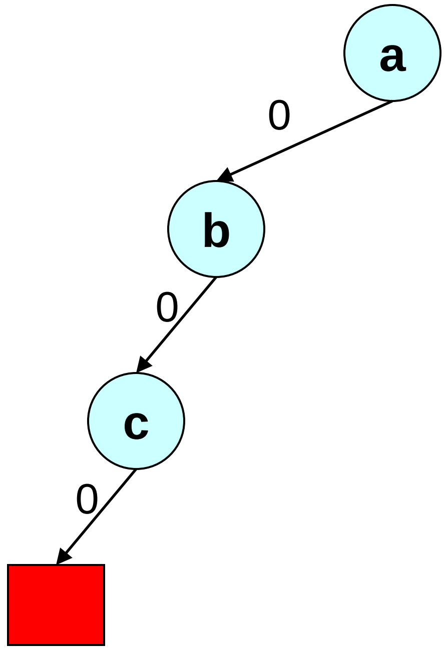
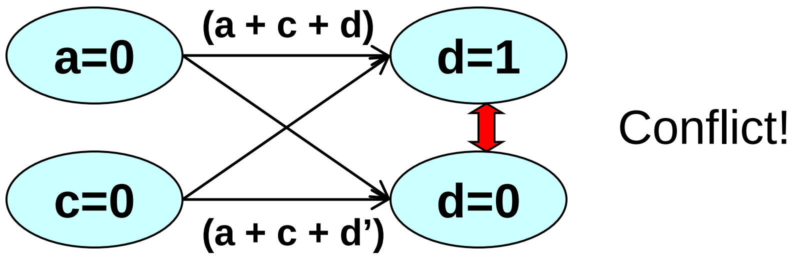


Implication Graph



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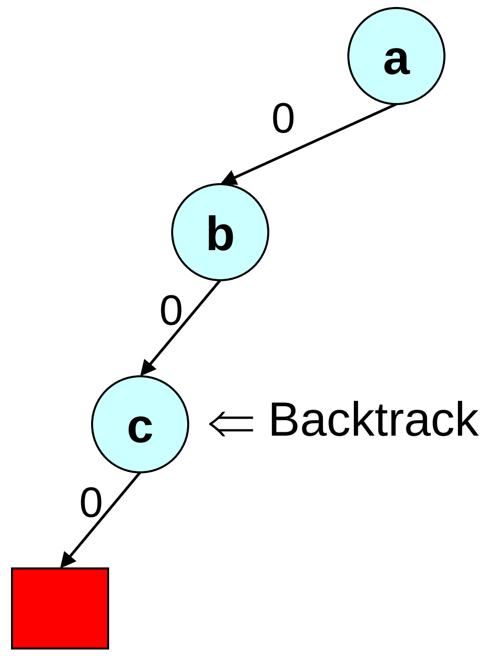

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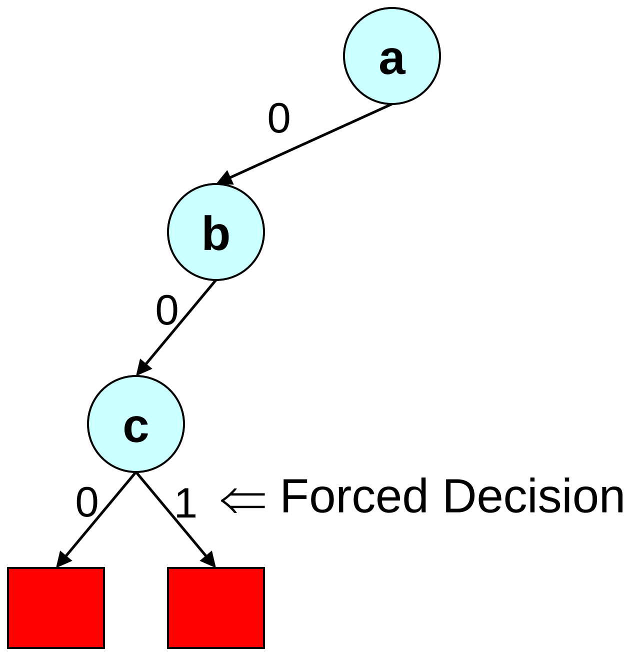

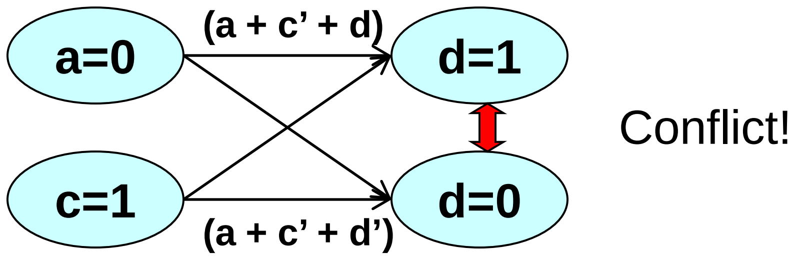

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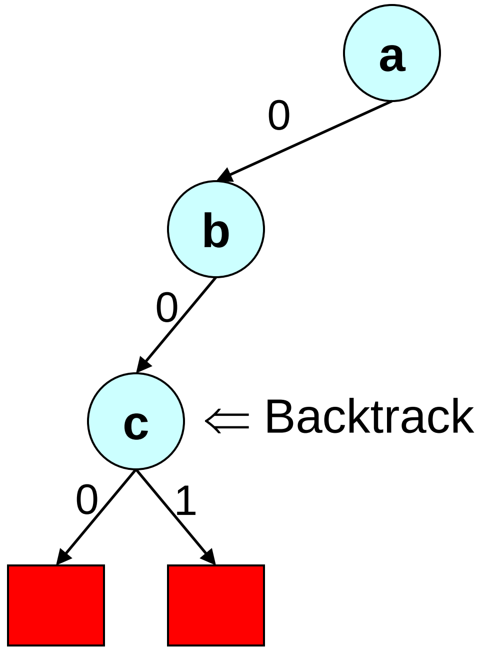

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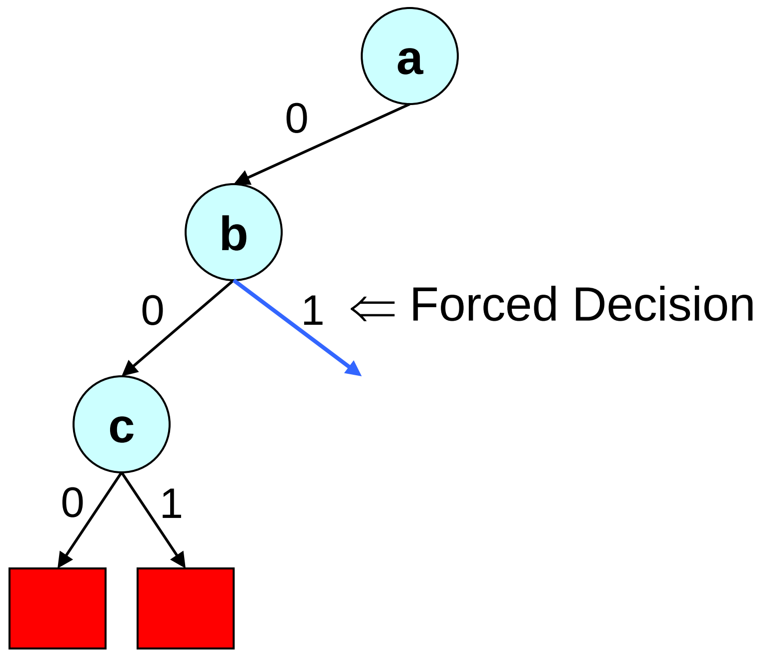

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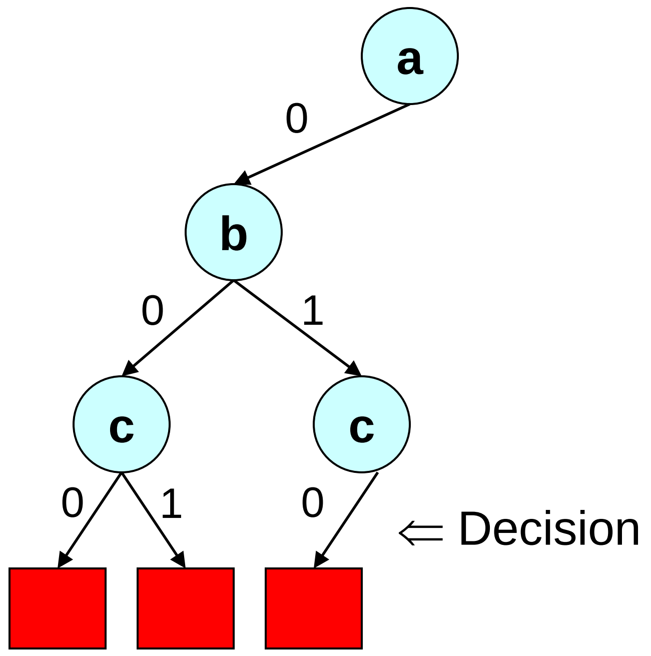

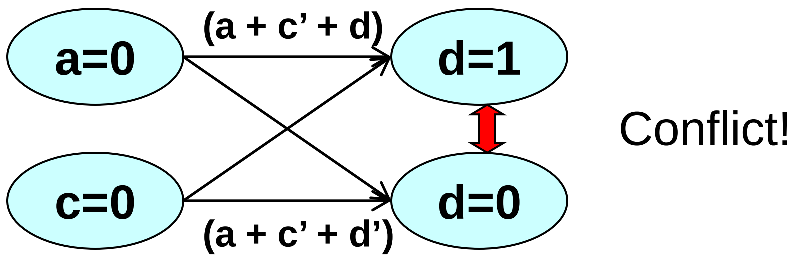

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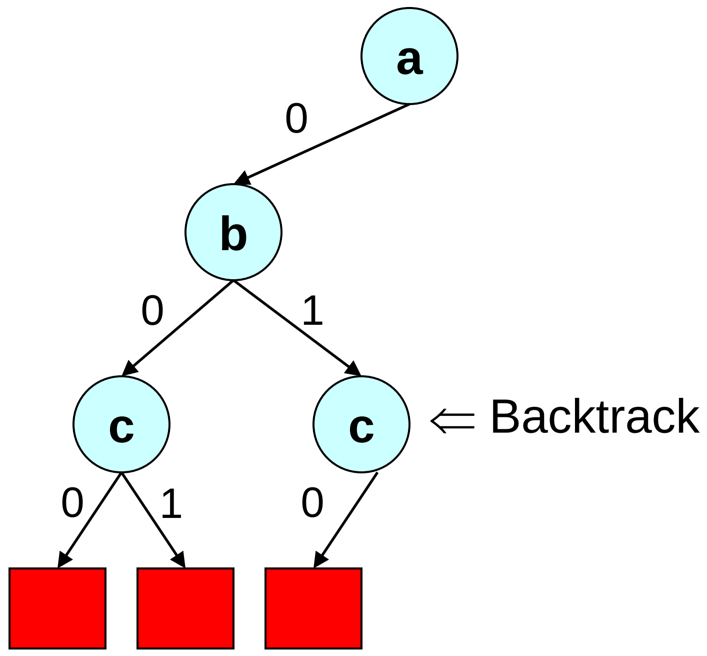

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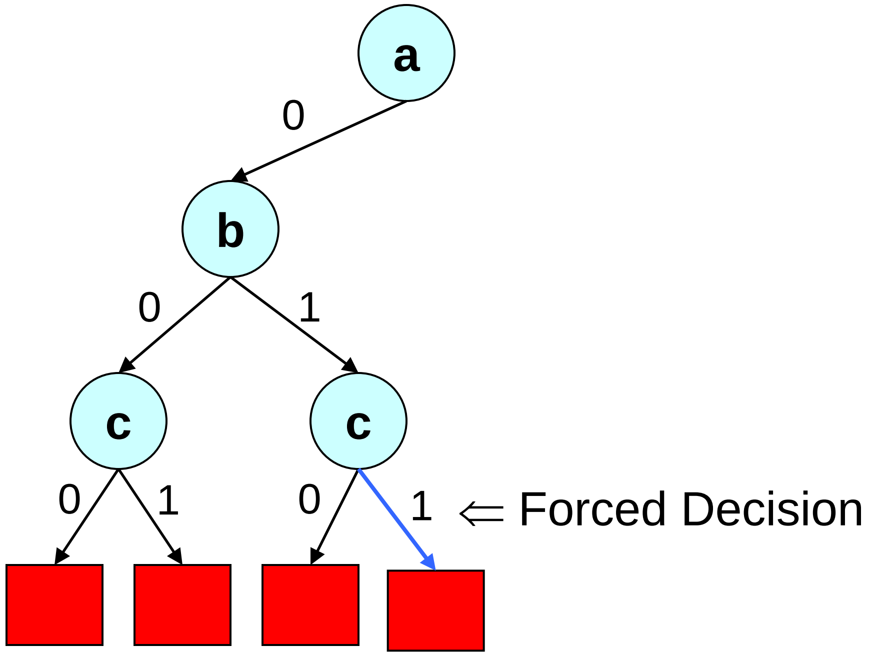

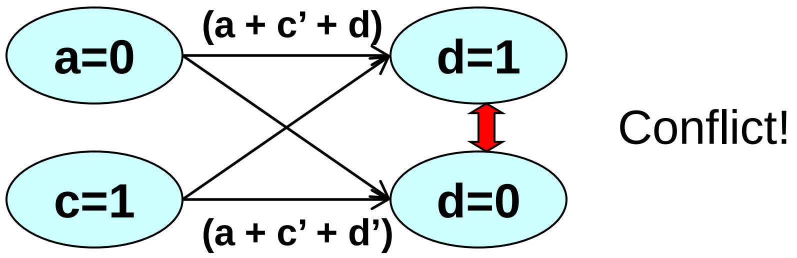

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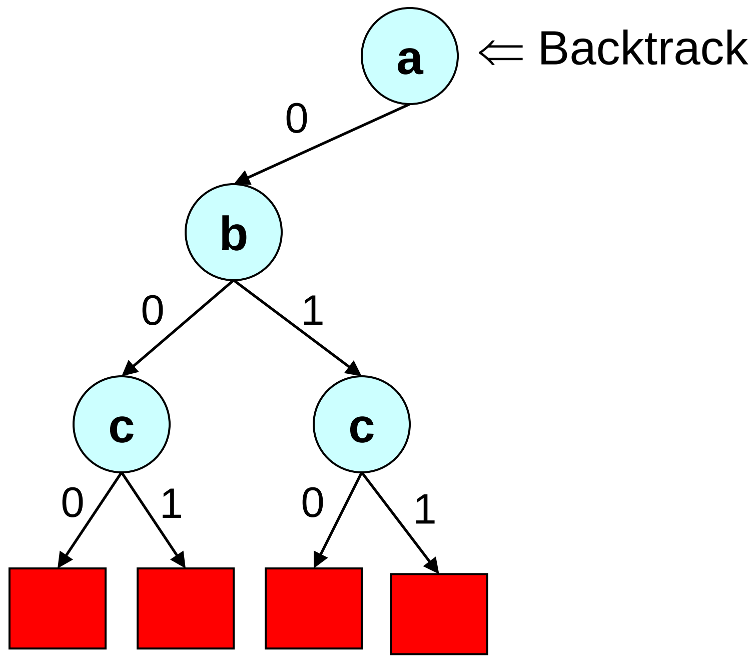

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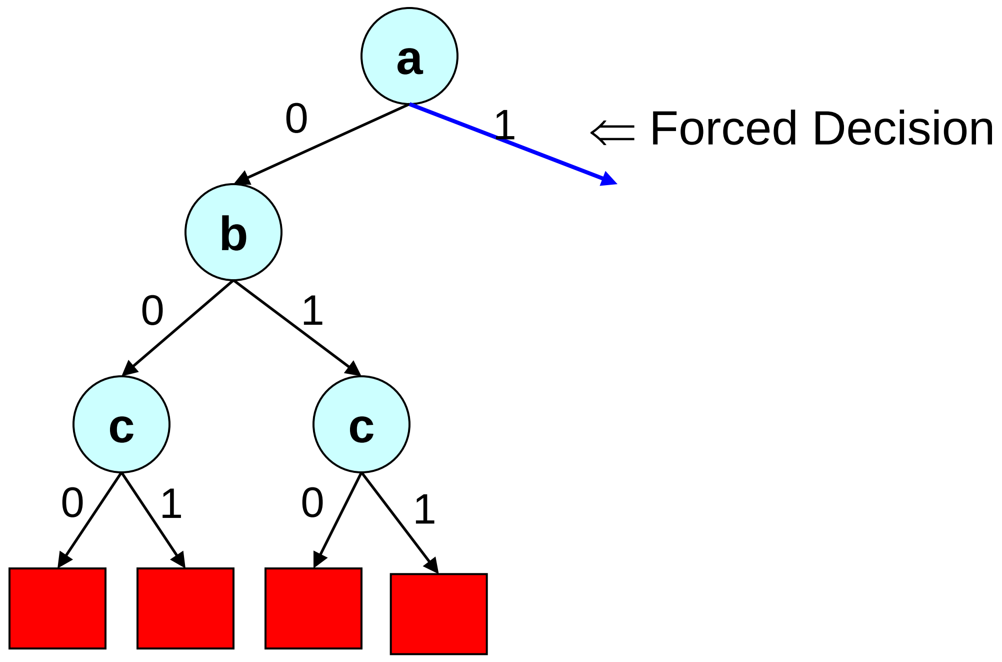

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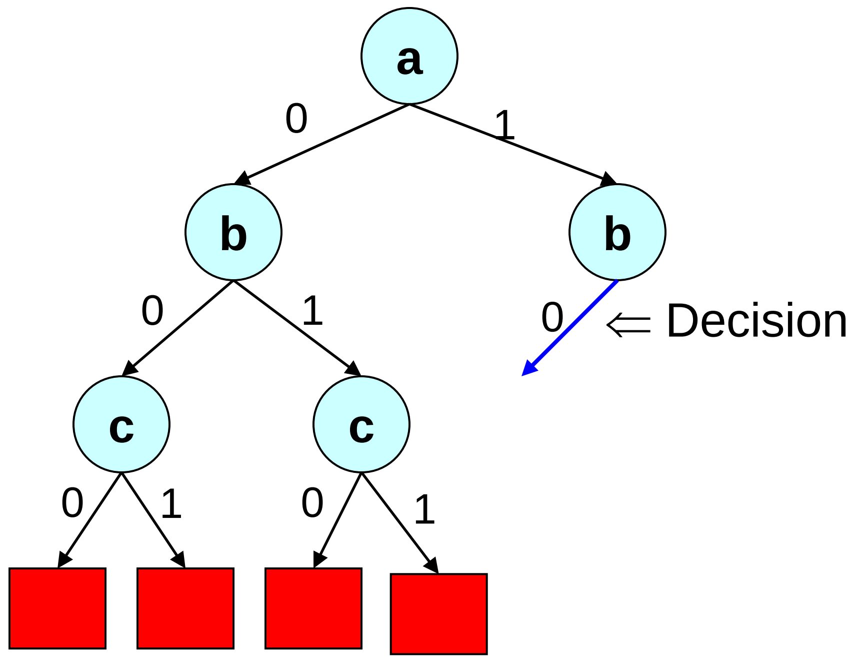

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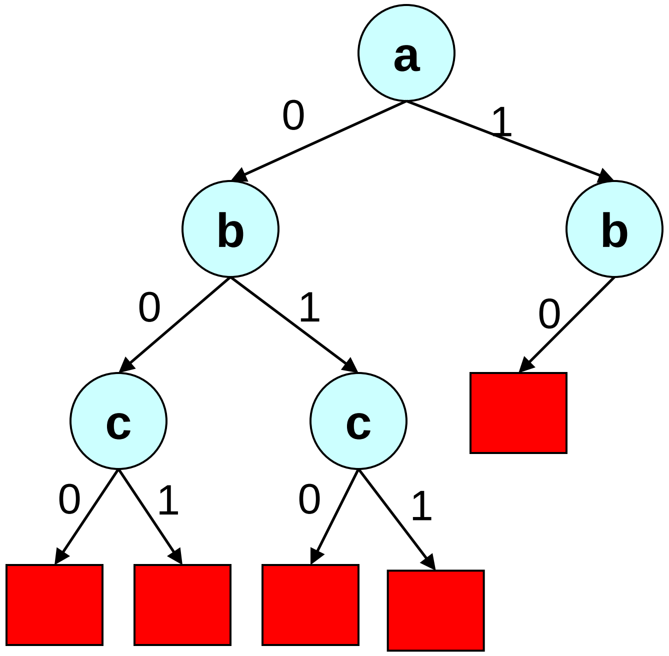

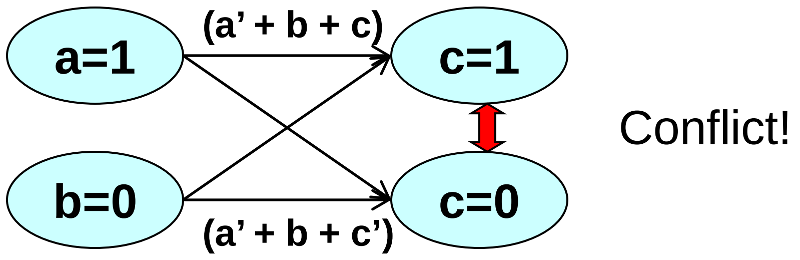

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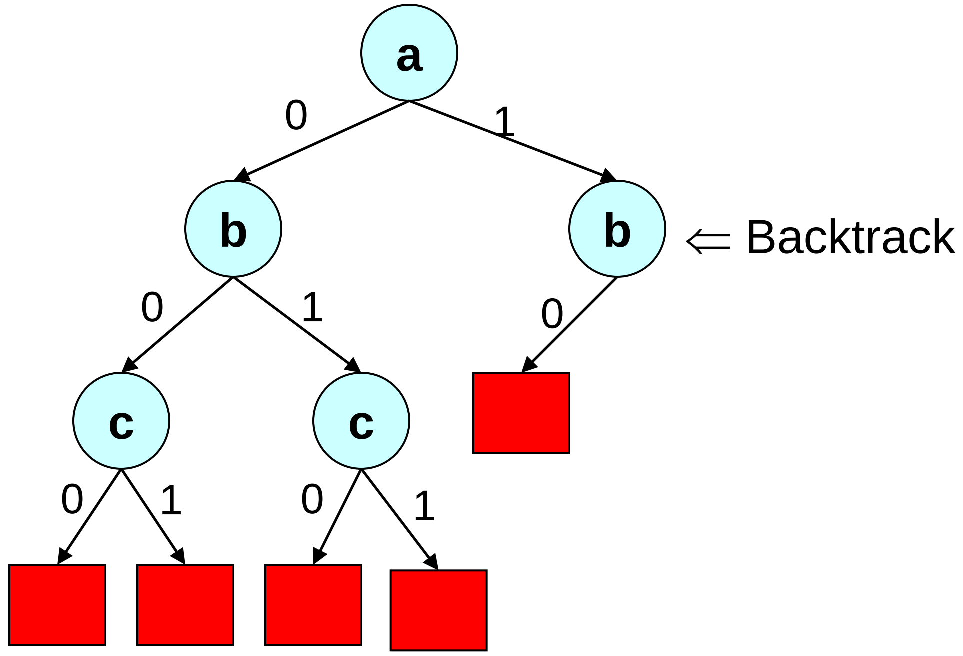

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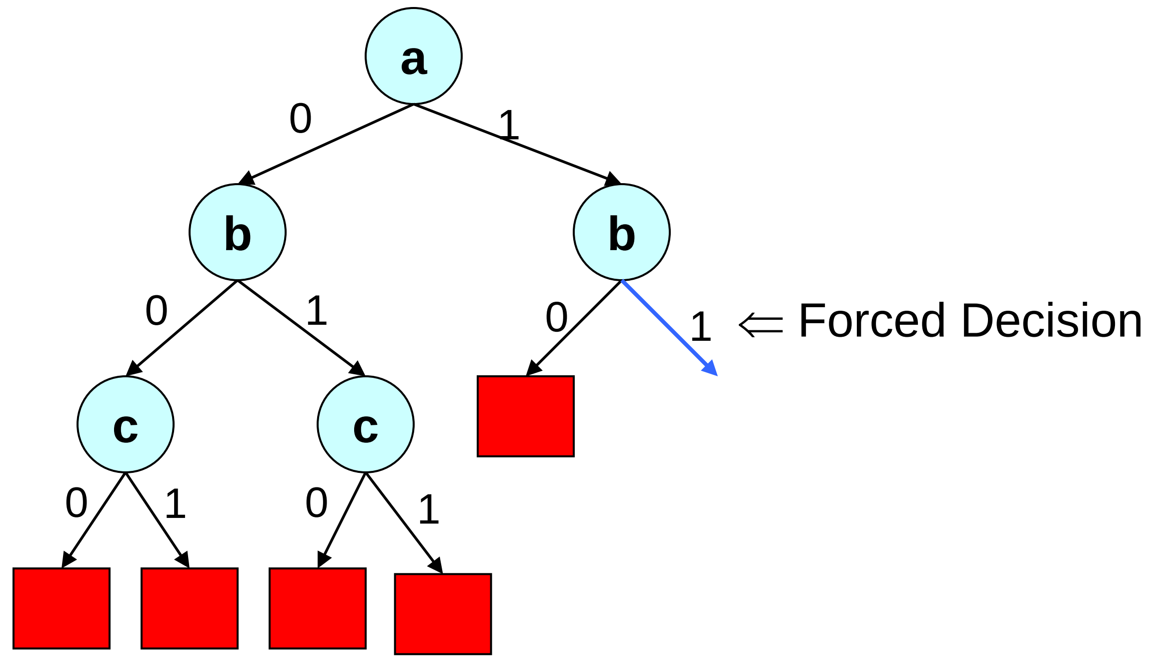

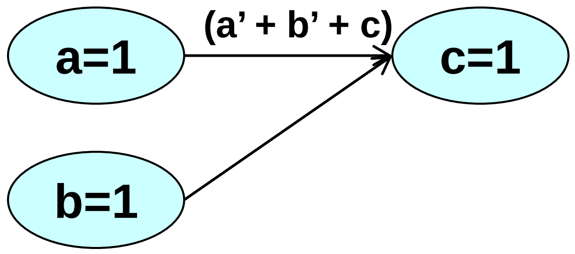

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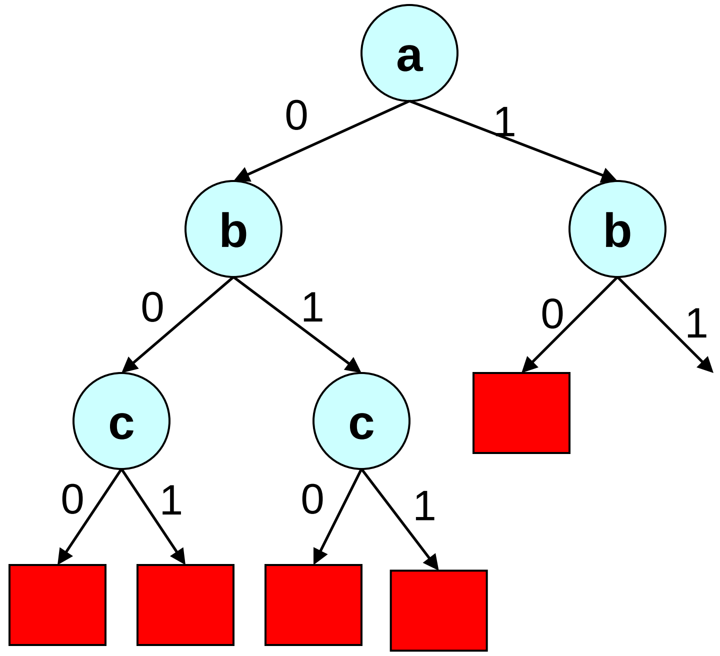

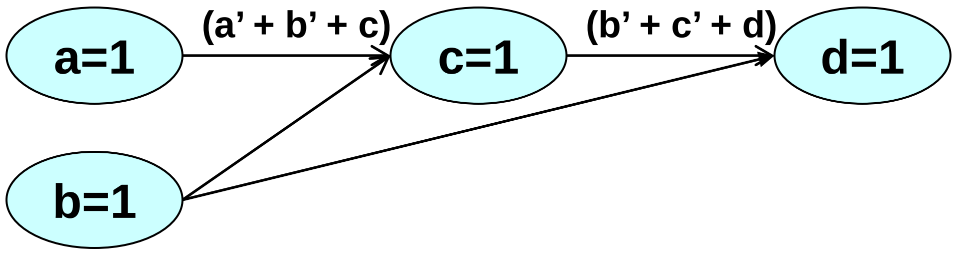

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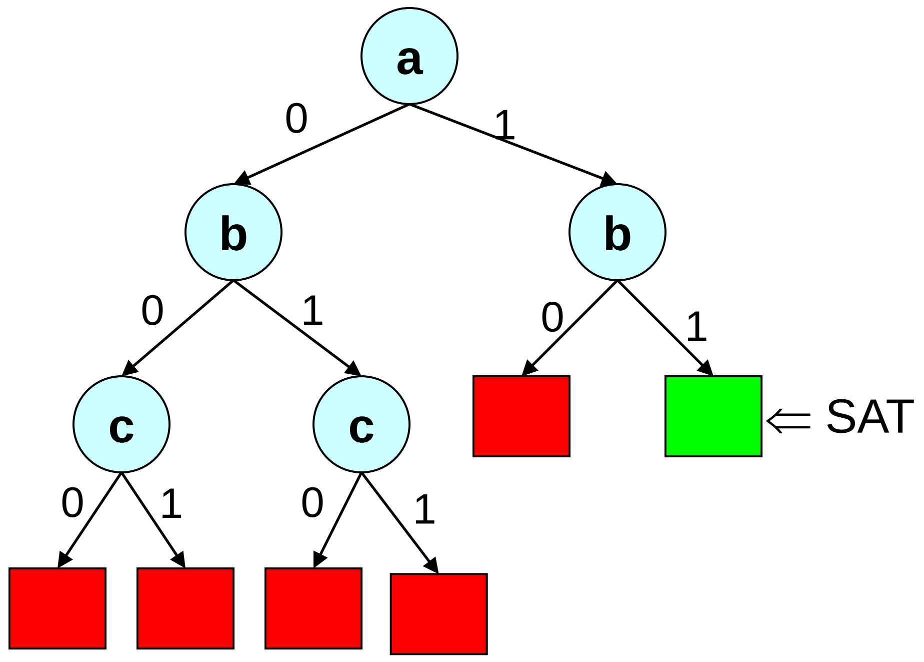
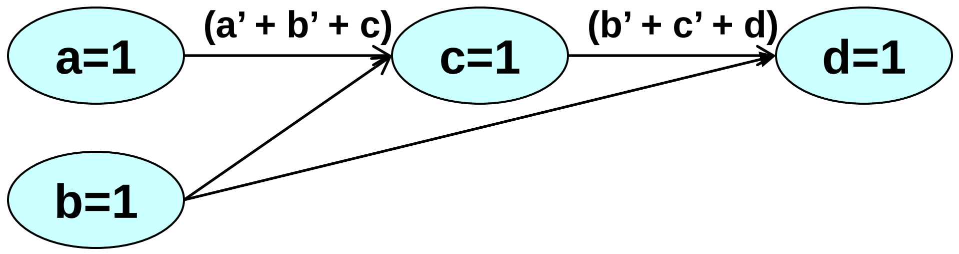




Implications and Boolean Constraint Propagation

- Implication
  - A variable is forced to be assigned to be True or False based on previous assignments.
- Unit clause rule (rule for elimination of one literal clauses)
  - An unsatisfied clause is a unit clause if it has exactly one unassigned literal.

$$(a + b' + c)(b + c')(a' + c')$$

a = T, b = T, c is unassigned

Satisfied Literal

Unsatisfied Literal

Unassigned Literal

  - The unassigned literal is implied because of the unit clause.
- Boolean Constraint Propagation (BCP)
  - Iteratively apply the unit clause rule until there is no unit clause available.
- Workhorse of DLL based algorithms – used to reduce the number of decisions made

Features of DLL

- Eliminates the exponential memory requirements of DP
- Exponential time is still a problem
- Limited practical applicability – largest use seen in automatic theorem proving
- Very limited size of problems (partly due to capacity of computers at the time)

Exercise

- Apply the DLL algorithm without and with the use of implications to determine the satisfiability of the following formula:

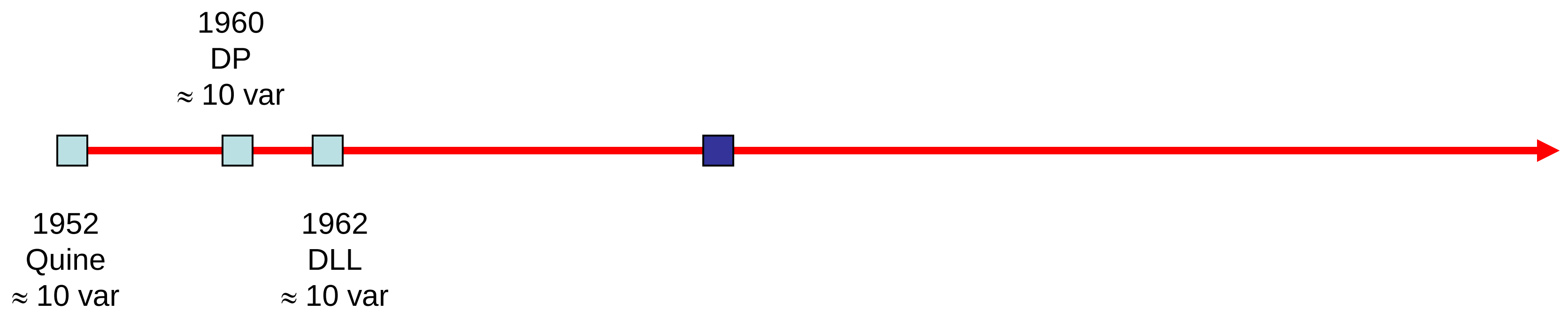
$$(a+b+c)(a+b+c')(a'+b+c')(a+c+d)(a'+c+d)(a'+c+d')(b'+c'+d')(b'+c'+d)$$

General Principles

- Backtracking can be an effective approach to solving search problems by systematically exploring the solution space
- Use of implications significantly reduces search space
  - No need to backtrack on decisions that are forced by implications

The Timeline

1986
Binary Decision Diagrams (BDDs)
≈100 var



R. Bryant. “Graph-based algorithms for Boolean function manipulation”. *IEEE Trans. on Computers*, C-35, 8:677-691, 1986.

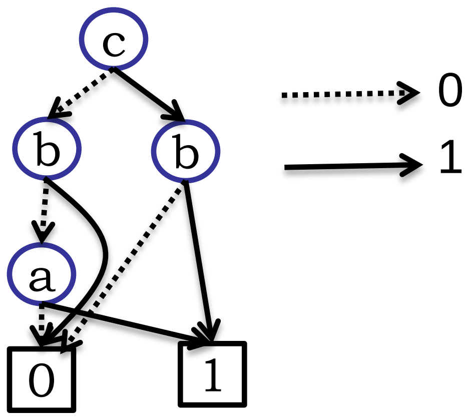
Binary Decision Diagram (BDD)

- Binary Decision Tree has large number of nodes
- Key idea: Share sub-trees and eliminate redundancy to reduce size
- More about BDDs later in the class

Example:

a b c	f
0 0 0	0
0 0 1	0
0 1 0	0
0 1 1	1
1 0 0	1
1 0 1	0
1 1 0	0
1 1 1	1

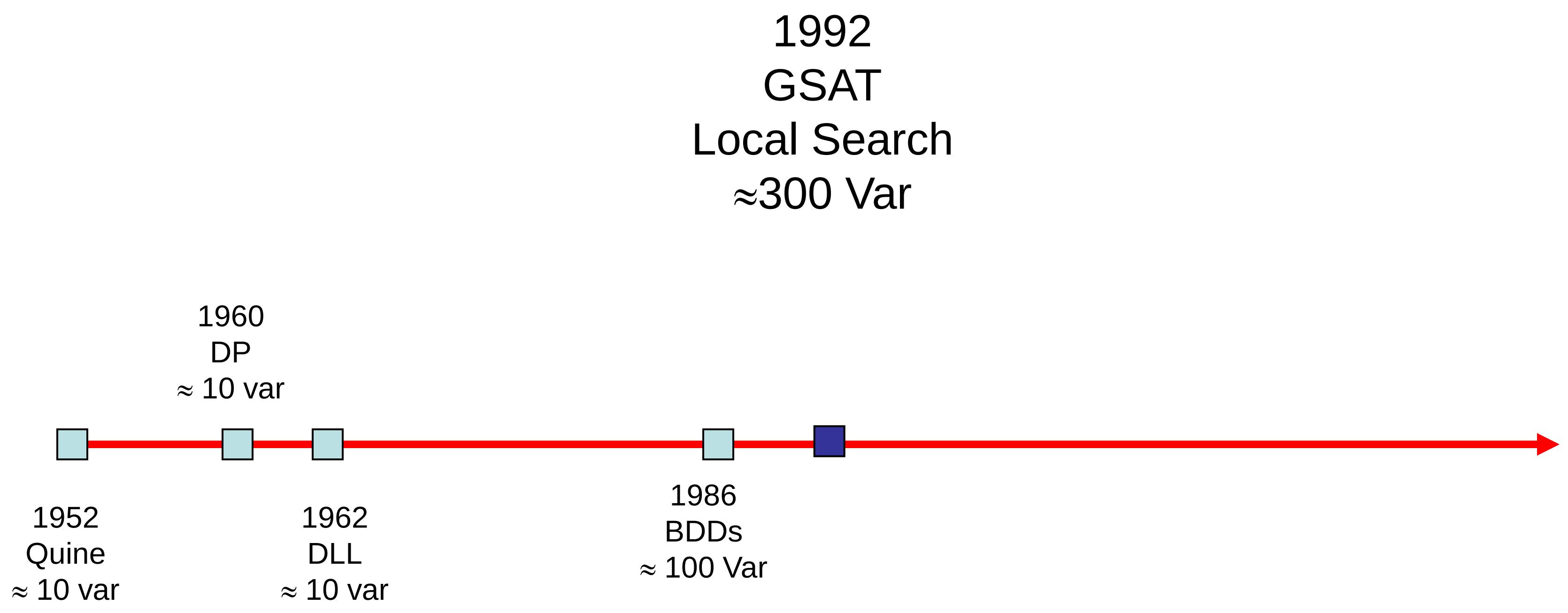
Binary Decision Diagram:



Using BDDs to Solve SAT

- Store the formula in a Reduced Ordered BDD (ROBDD) representation.
  - Compacted form of the binary decision tree.
- Construction rules guarantee canonicity
  - Only one way to represent constant 0 (unsatisfiable) function!
  - Any other function is satisfiable
- Overkill for SAT.

The Timeline

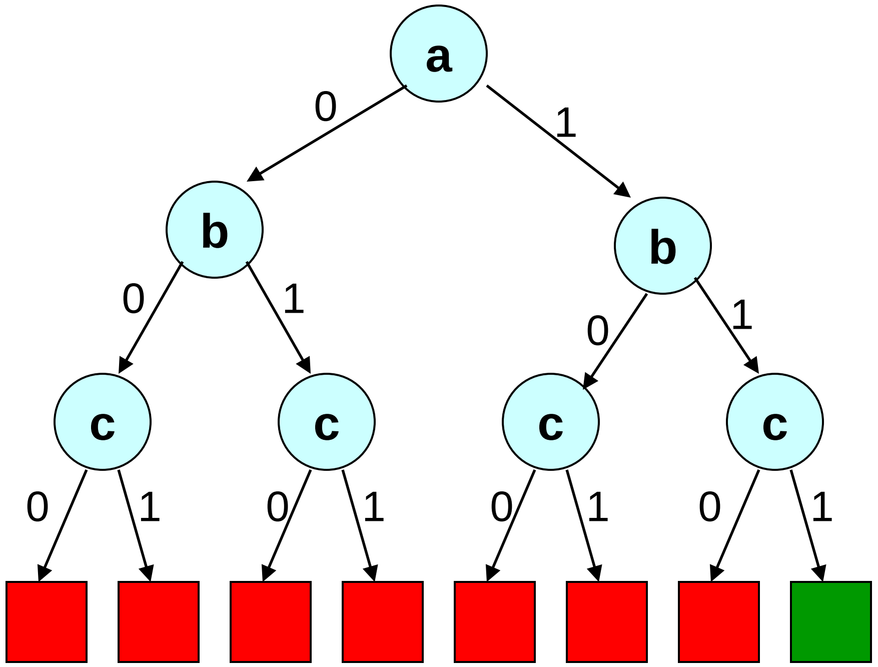


B. Selman, H. Levesque, and D. Mitchell. “A new method for solving hard satisfiability problems,” *Proc. National Conference on Artificial Intelligence (AAAI)*, 1992.

B. Selman and H. Kautz “Noise Strategies for Improving Local Search,” *Proc. National Conference on Artificial Intelligence (AAAI)*, 1994.

Local Search

(a + b)
(a + b’)
(a’ + b)
(a’ + b’ + c)
(b + c)



Local Search

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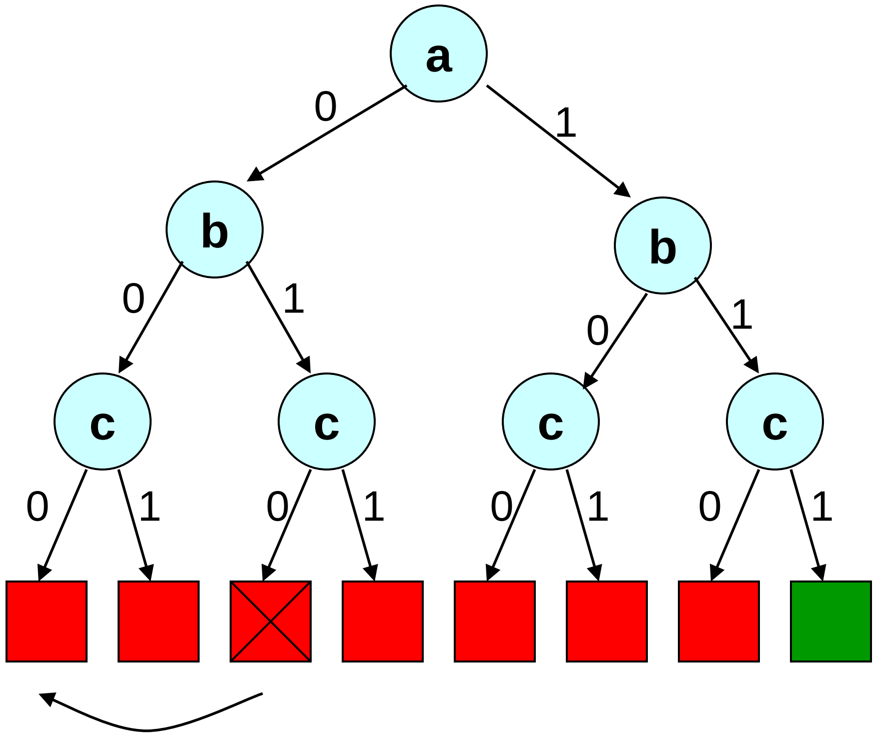


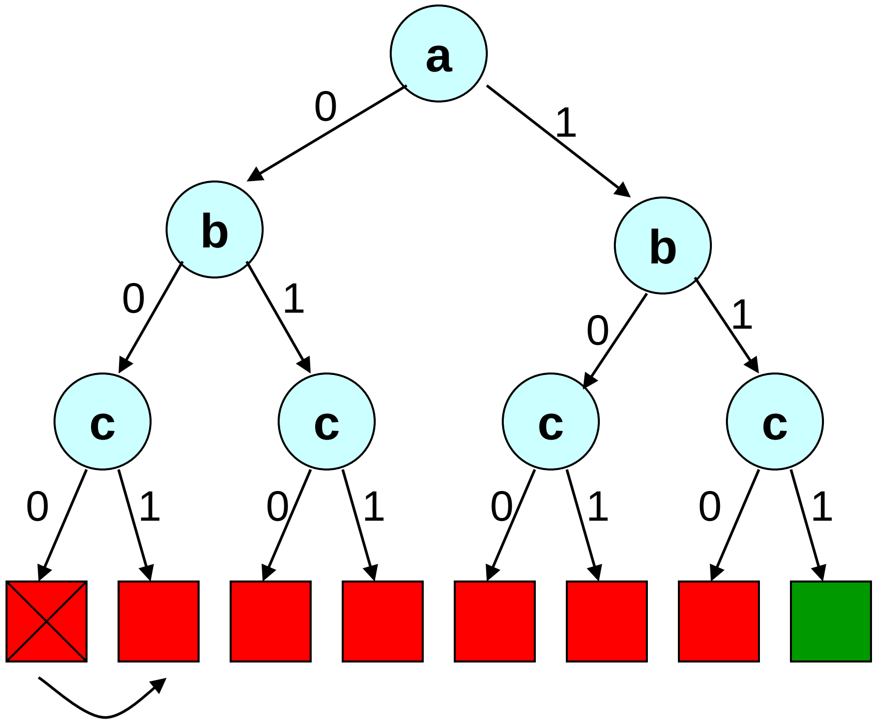
Cost Function (CF):

# of un-satisfied clauses:

CF(a=0,b=1,c=0) = 1

Local Search

(a + b)
(a + b')
(a' + b)
(a' + b' + c)
(b + c)



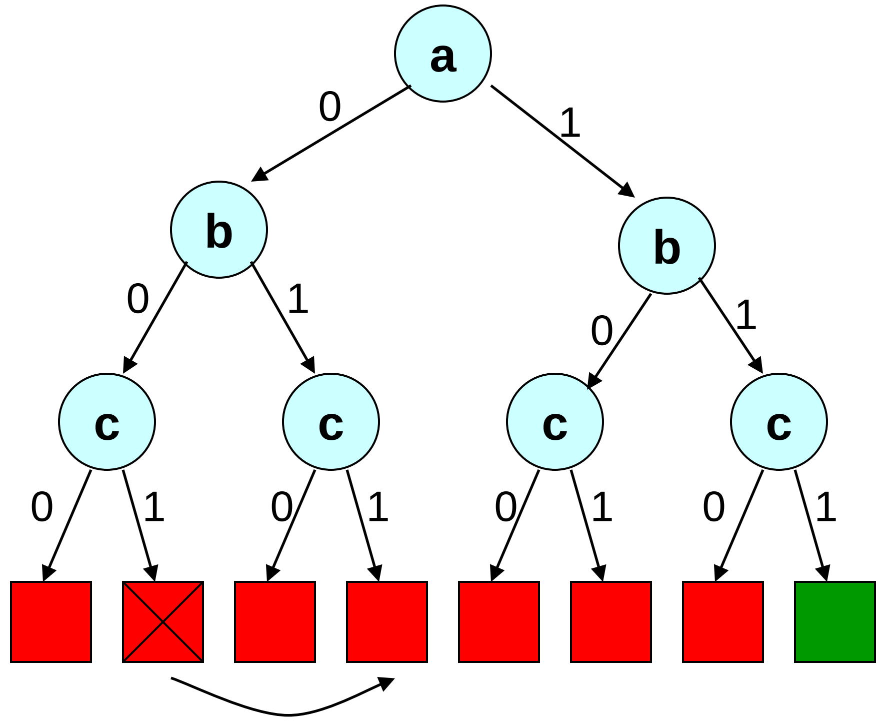
Cost Function F:

# of un-satisfied clauses:

CF(a=0,b=0,c=0) = 2

Local Search

(a + b)
(a + b')
(a' + b)
(a' + b' + c)
(b + c)



Cost Function F:

# of un-satisfied clauses:

CF(a=0,b=0,c=1) = 1

Local Search

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(a + b')
(a' + b)
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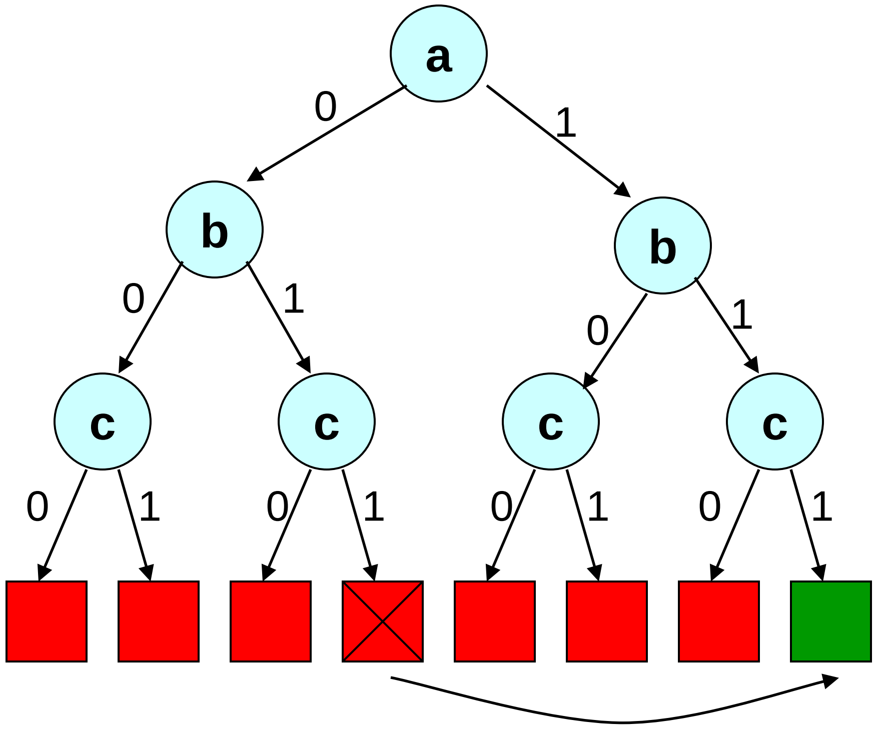

Cost Function F:

# of un-satisfied clauses:

CF(a=0,b=1,c=1) = 1

Local Search

(a + b)
(a + b')
(a' + b)
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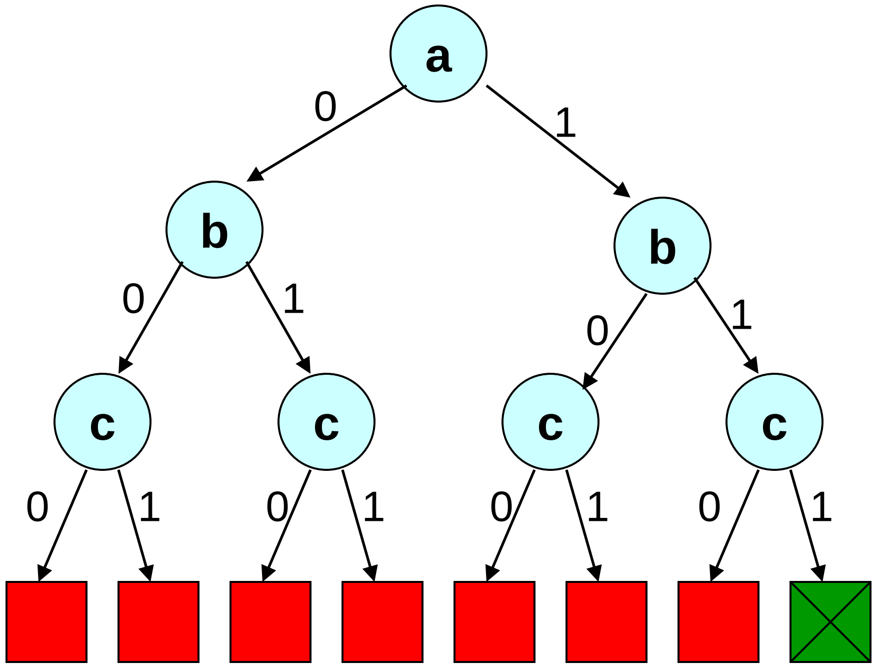


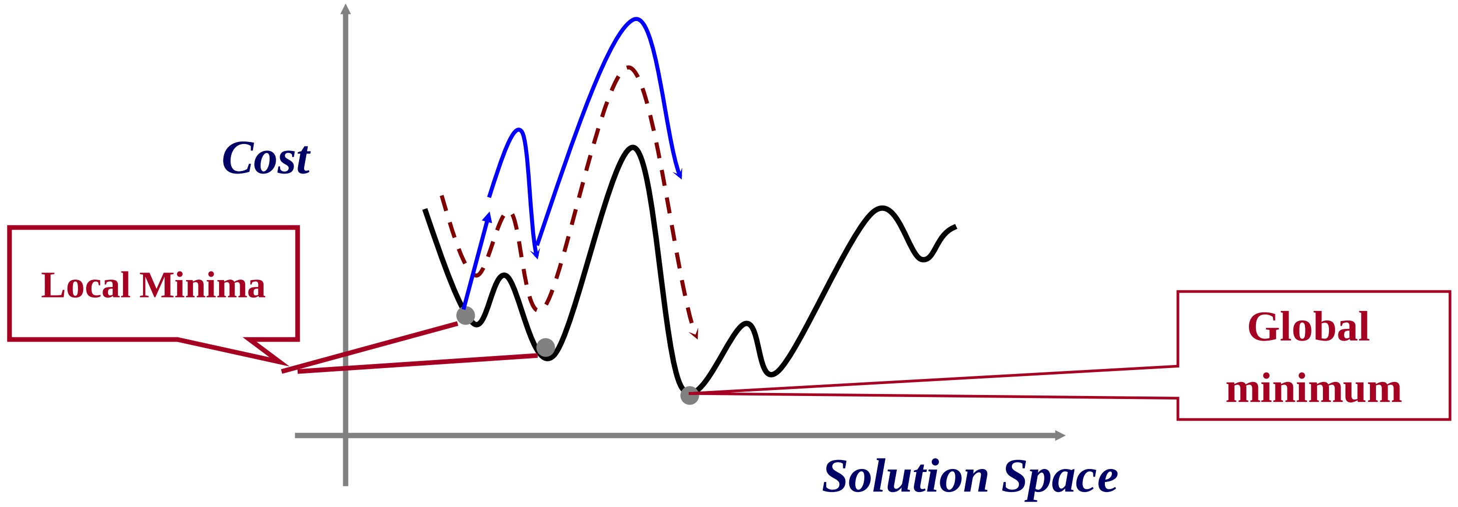
Cost Function F:

# of un-satisfied clauses:

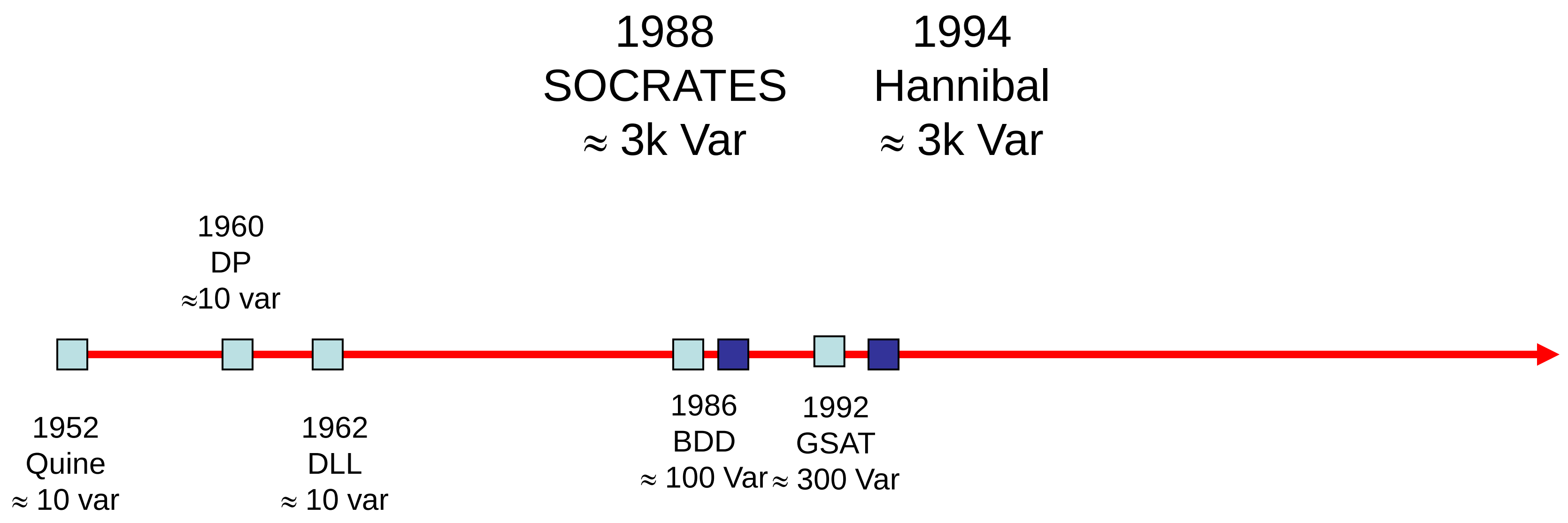
CF(a=1,b=1,c=1) = 0

Local Search (GSAT, WSAT)

- Hill climbing algorithms
- Make short local moves
- Probabilistically accept moves that worsen the cost function to enable exits from local minima
- Incomplete SAT solvers
  - Geared towards satisfiable instances, cannot prove unsatisfiability



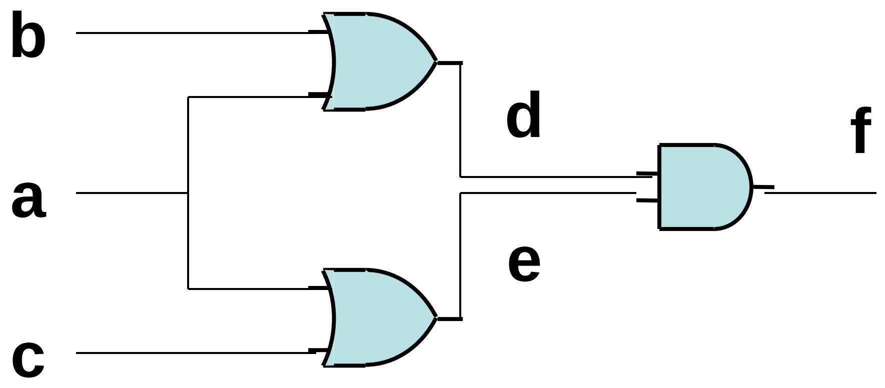
The Timeline



EDA Drivers (ATPG, Equivalence Checking) start the push for practically useable algorithms!
Deemphasize random/synthetic benchmarks.

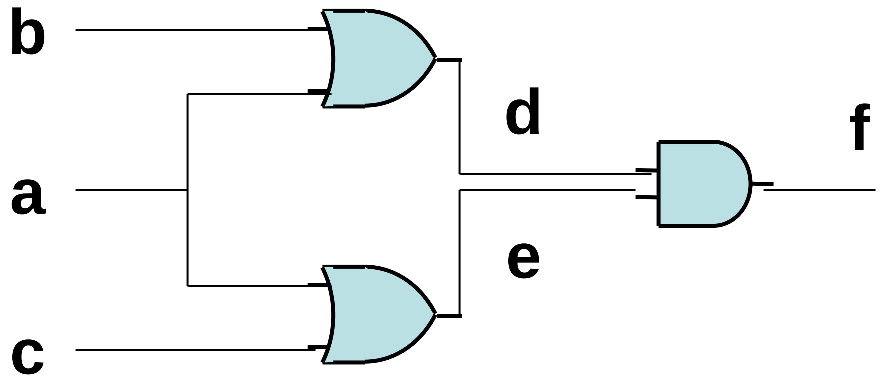
SOCRATES

M. H. Schulz, E. Auth, “SOCRATES: A highly efficient automatic test pattern generation system”, *IEEE Trans. Computers* C-37, 7:126-137, 1988



$a=1 \Rightarrow f=1$

SOCRATES

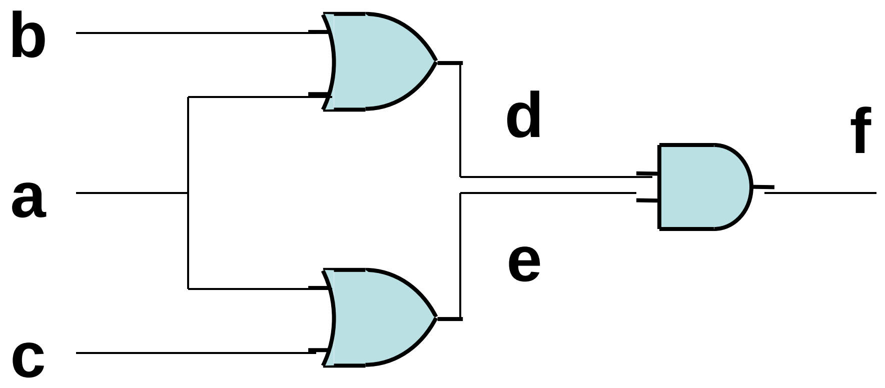


f=0⇒a=0

- Key idea : Use contra-positive for learning invariants
- Application to SAT : Add new clauses to the CNF, limits search space through faster backtrack

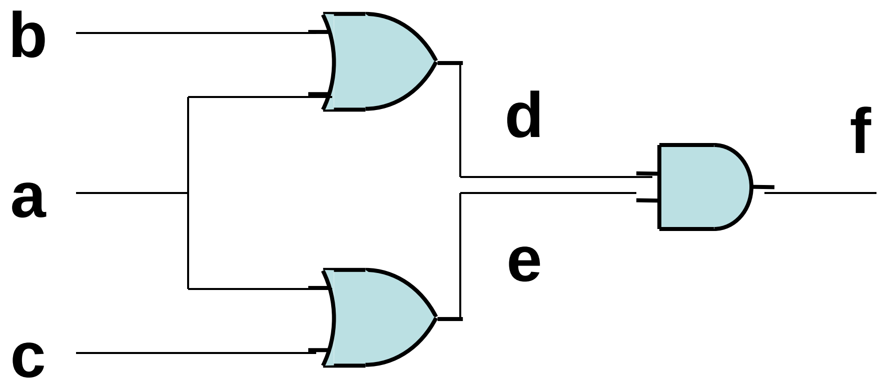
Hannibal

W. Kunz, “HANNIBAL: An efficient tool for logic verification based on recursive learning”, *Proc. ICCAD*, 1993



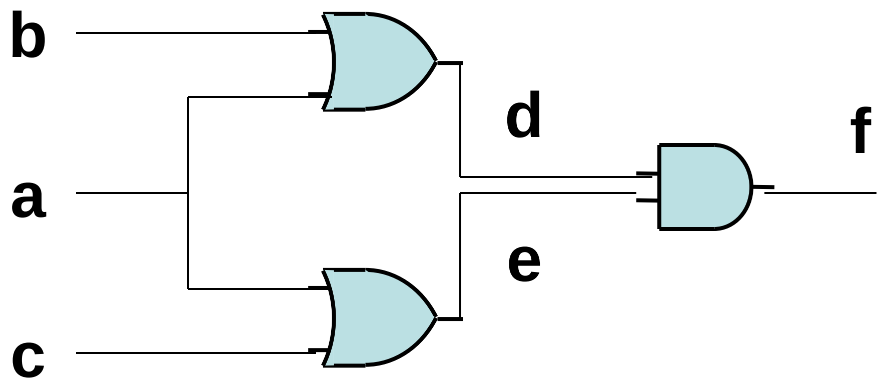
f=0⇒d=0 or e=0

Hannibal



Try d=0, Then a=0 and b=0

Hannibal

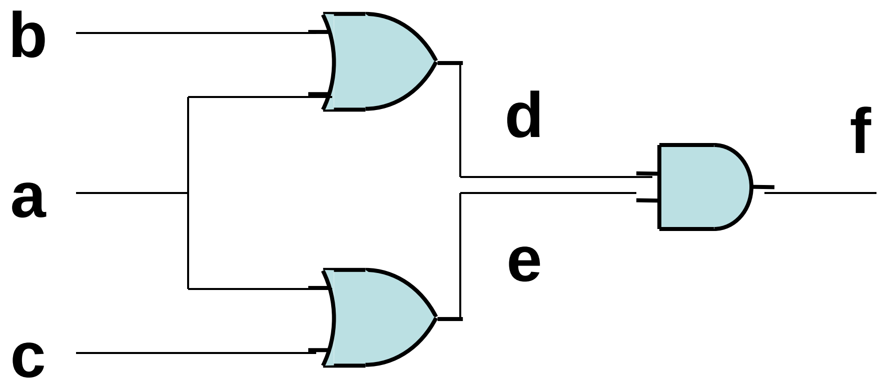


Try e=0, Then a=0 and c=0

Hannibal

Try d=0, Then a=0 and b=0

Try e=0, Then a=0 and c=0

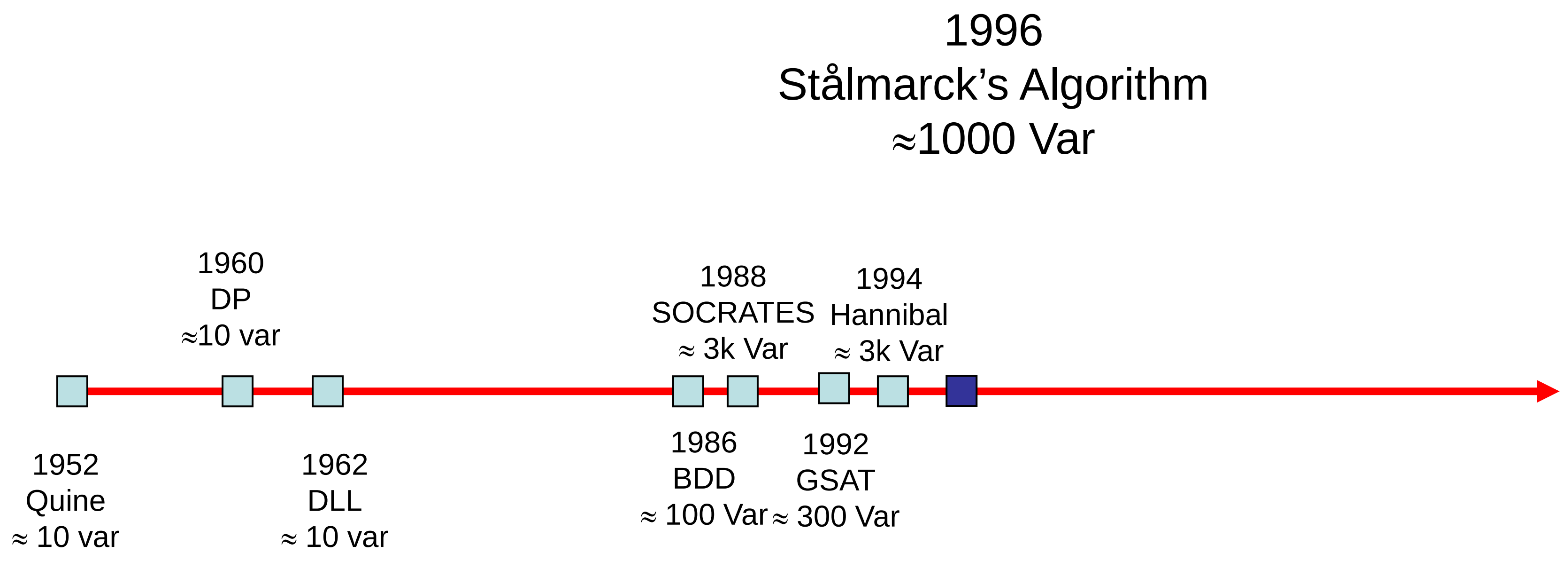


In both cases, a=0, so f=0 ⇒a=0

EDA Drivers

- Advances in ATPG and equivalence checking drove developments in SAT solvers
  - SOCRATES first incorporated learning
    - If P⇒Q, then Q’⇒P’
  - Hannibal used Recursive Learning with certain recursion depth
  - SOCRATES, Hannibal were able to handle practical sized circuits
- Use circuit specific information, so cannot immediately generalize this to arbitrary formulae

The Timeline



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M. Sheeran and G. Stålmarck "A tutorial on Stålmarck's proof procedure", *Proc. FMCAD*, 1998

Stålmarck's algorithm

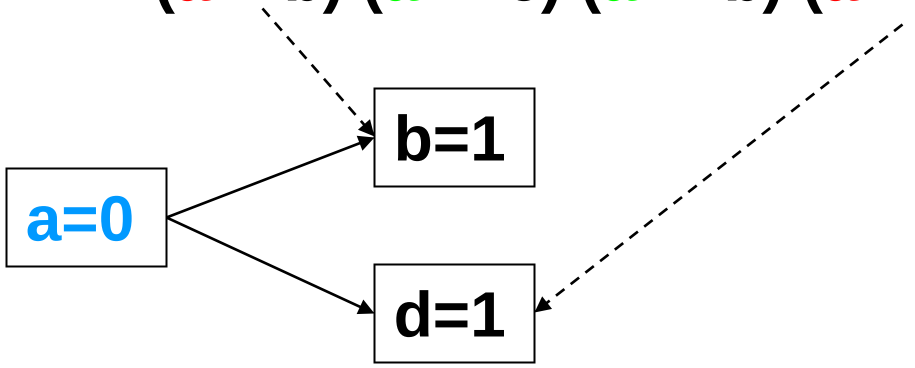
- Try both sides of a branch to find forced decisions (relationships between variables)

(a + b) (a' + c) (a' + b) (a + d)

Stålmarck's algorithm

- Try both sides of a branch to find forced decisions

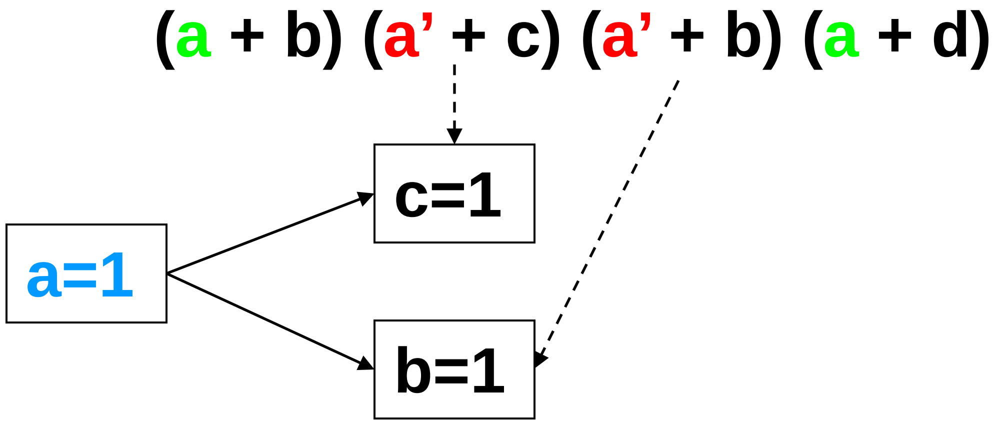
$$(a + b)\,(a' + c)\,(a' + b)\,(a + d)$$



$$\boxed{a=0 \Rightarrow b=1, d=1}$$

Stålmarck's algorithm

- Try both sides of a branch to find forced decisions



$\boxed{a=0 \Rightarrow b=1, d=1}$

$\boxed{a=1 \Rightarrow b=1, c=1}$

Stålmarck's algorithm

- Try both sides of a branch to find forced decisions

(a + b) (a' + c) (a' + b) (a + d)

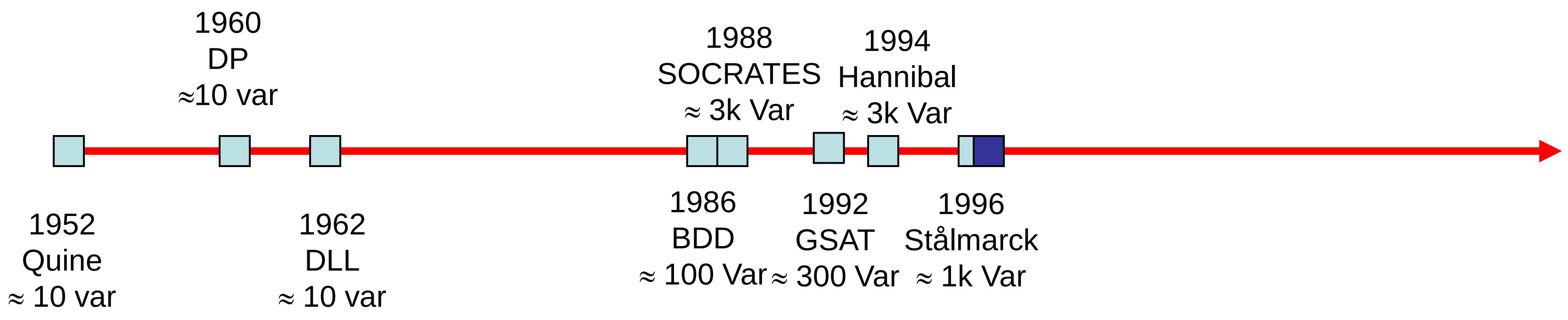
$$\boxed{a=0 \Rightarrow b=1, d=1} \quad \Rightarrow \; b=1$$

$$\boxed{a=1 \Rightarrow b=1, c=1}$$

- Repeat for all variables
- Repeat for all pairs, triples,… till either SAT or UNSAT is proved

The Timeline

1996
GRASP
Conflict Driven Learning,
Non-chornological Backtracking
≈1k Var



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J. P. Marques-Silva and Karem A. Sakallah, “GRASP: A Search Algorithm for Propositional Satisfiability”, *IEEE Trans. Computers*, C-48, 5:506-521, 1999.