ECE 595, Section 10
Numerical Simulations
Lecture 12: Applications of FFT

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Outline

• Recap from Friday
• Real FFTs
• Multidimensional FFTs
• Applications:
  – Correlation measurements
  – Filter diagonalization method
Recap from Friday

• Recap from Wednesday
• Fourier Analysis
  – Scalings and Symmetries
  – Sampling Theorem
• Discrete Fourier Transforms
  – Naïve approach
  – Danielson-Lanczos lemma
  – Cooley-Tukey algorithm
Real FFTs

• For real functions, the general complex FFT procedure is wasteful

• Solutions:
  – Pack twice as many FFTs into each calculation
  – Reduce length by half, sort out result
  – Use sine and cosine transforms

• Application: signal processing of experimental measurement data
Multidimensional FFTs

- Applications: image processing, band structures
- Definition:

\[
F(n_x, n_y) = \sum_{k_x=1}^{N} \sum_{k_y=1}^{N} f_{k_x k_y} e^{2\pi j \left[ k_x n_x + k_y n_y \right]/N}
\]

- For FFT data in 2D or 3D, can efficiently perform FT in each dimension successively

Correlation Measurements

• Application: ultrafast optics, quantum optics

• Correlation for discrete data defined by:

\[ g_2(m) = \sum_{k=1}^{N} f_k h_{k+m} \]

• Autocorrelation: special case where \( f = h \)

Correlation Measurements

• Autocorrelation powerful signature of the nature of one’s data set
• Largest value for \( m=0 \)
• Pure noise: \( \delta \)-function correlated
• Pure periodic signal: cross-correlation also has same period
• Most signals decay with characteristic correlation time \( \tau_c \)
Time-domain data analysis

• Many PDE solvers produce a time series of data warranting spectral analysis
• Examples: finite-difference time domain, drift-diffusion models
Signal Processing

• Most obvious approach: least-squares fit to FFT of time-series data

• Given a set of narrow Lorentzian peaks, should fit well, right? Problem solved!
Signal Processing

• But what if the decay is slow, and unfinished?
• The FFT of the time-series will look significantly different from goal
Signal Processing

• An even greater challenge – what if you have two time decays with relatively close frequencies (this case is fairly common)?

• Can’t even detect the number of modes!
Signal Processing

• Need to find an alternative strategy to straightforward FFTs
• Might want to add damping explicitly
• Most obvious approach known as decimated signal diagonalization
• One particularly useful approach devised by Mandelshtam is known as filter diagonalization
Filter Diagonalization Method


Given time series $y_n$, write:

$$y_n = y(n\Delta t) = \sum_k a_k e^{-i\omega_k n\Delta t}$$

...find complex amplitudes $a_k$ & frequencies $\omega_k$ by a simple linear-algebra problem!

Idea: pretend $y(t)$ is autocorrelation of a quantum system:

$$\hat{H} \psi \rangle = i \frac{\partial}{\partial t} \psi \rangle$$

time-$\Delta t$ evolution-operator:

$$\hat{U} = e^{-i\hat{H} \Delta t}$$

say:

$$y_n = \langle \psi(0) | \psi(n\Delta t) \rangle = \langle \psi(0) | \hat{U}^n | \psi(0) \rangle$$
Filter-Diagonalization Method

\[ y_n = \langle \psi(0) | \psi(n\Delta t) \rangle = \langle \psi(0) | \hat{U}^n | \psi(0) \rangle \]

\[ \hat{U} = e^{-i\hat{H}\Delta t} \]

We want to diagonalize \( U \): eigenvalues of \( U \) are \( e^{i\omega\Delta t} \)

...expand \( U \) in basis of \( |\psi(n\Delta t)\rangle \):

\[ U_{m,n} = \langle \psi(m\Delta t) | \hat{U} | \psi(n\Delta t) \rangle = \langle \psi(0) | \hat{U}^m \hat{U} \hat{U}^n | \psi(0) \rangle = y_{m+n+1} \]

\( U_{mn} \) given by \( y_n \)'s — just diagonalize known matrix!
Filter-Diagonalization Summary


$U_{mn}$ given by $y_n$’s — just diagonalize known matrix!

A few omitted steps:

— Generalized eigenvalue problem (basis not orthogonal)
— Filter $y_n$’s (Fourier transform):
  small bandwidth = smaller matrix (less singular)

• resolves many peaks at once
• # peaks not known a priori
• resolve overlapping peaks
• resolution >> Fourier uncertainty
Next Class

• Is on Friday, Feb. 8
• Will discuss FFTW