Outline

• Recap from Monday
• Perfectly Matched Layers
• Finite Elements
• Finite Element BPM
• Reducing FEM Errors
Recap from Monday

• Derivation of Beam Propagation Method
• Nonlinear Schrodinger equation
• Comparison of BPM Strategies
  – FFT
  – Uniform spatial grid
  – Finite element
Recap from Monday

• Beam propagation amounts to solving:

\[ \frac{\partial \phi}{\partial z} = (U + W)\phi \]

where:

\[ U = \frac{j}{2\beta} \nabla_\perp^2 \]

\[ W = \frac{j k_\perp^2}{2\beta} \]
Perfectly Matched Layers

- In order to prevent lateral reflections (e.g., from PEC boundaries), can introduce perfectly matched layers (PML)
- Several formulations (including split-field and uniaxial), but here we’ll follow stretched coordinate PML
- Effected by the transformation:

\[ \nabla \rightarrow A \cdot \nabla \]

\[ A = \begin{pmatrix} 1 - j\beta & 0 & 0 \\ 0 & 1 - j\beta & 0 \\ 0 & 0 & 1 \end{pmatrix} \]

\[ \beta = -\frac{3\lambda\rho^2}{4\pi \eta d^3} \ln R \]
Perfectly Matched Layers

- Residual reflection scales as a power law with PML thickness
- Cubic absorption increase with position offers the best performance

Finite Elements

• Shapes: 1D, 2D, and 3D

• Shape functions:

1D: \( u(x) = \alpha + \beta x + \gamma x^2 + \cdots \)

2D/3D: \( u(x) = \sum_{k=0}^{d} [\alpha_k x^k + \beta_k y^k + \gamma_k z^k] \)
Finite Elements

• Lagrange functions:
  \[ \lambda_0(x) = \frac{\xi_1 - x}{\xi_1 - \xi_o} \]
  \[ \lambda_1(x) = \frac{x - \xi_o}{\xi_1 - \xi_o} \]

Basis functions \( \varphi_j(x) \) combine the Lagrange functions with compact support

Finite Element BPM

• In general, can formulate FE problems as:

\[ L u = b \]

– \( L \) is the stiffness matrix, representing overlap between basis functions
– \( b \) is the integral of given PDE with respect to basis
– \( u \) is unknown
Finite Element BPM

• Can define error function as:
  \[ E = Lu - b \]

• In order to eliminate errors, set weighted residual \( w_i \) in test space \( v \) to zero:
  \[ \int_v w_i (Lu - b) = 0 \]

• Galerkin’s method is a specific example of this:
  \[ \int_v \psi (Lu - b) = 0 \]

  where \( u(x) \) are the polynomials we saw earlier
Finite Element BPM

• Can refine accuracy of BPM for wide-angle beam propagation with second derivative in z:

\[
\frac{d\zeta}{dz} = -2j\beta\zeta - \nabla^2_\perp \phi - k^2_\perp \phi
\]

\[
\frac{d\phi}{dz} = \zeta
\]

• Can then choose a Padé approximant based on initial value of \(\zeta\). If \(\zeta(0) = 0\), then:

\[
\zeta = j\beta \left[ \sqrt{1 + \frac{\nabla^2_\perp + k^2_\perp}{\beta^2}} - 1 \right] \phi
\]
Finite Element BPM

• Applying Galerkin method to second-order BPM equations yields:

\[ h_T(x, y, z) = \sum_{j=1}^{N_{px}} h_{xj}(z)\psi_j(x, y)\hat{u}_x + \sum_{j=N_{px}+1}^{N_p} h_{yj}(z)\psi_j(x, y)\hat{u}_y \]

\[ [M] \frac{\partial^2 \{h_T\}}{\partial z^2} - 2\gamma [M] \frac{\partial \{h_T\}}{\partial z} + ([K] + \gamma^2 [M]) \{h_T\} = \{0\} \]

\[ [K]_{ij} = -\int_{\Omega} (\vec{k}_z \nabla_T \times \vec{\psi}_j, (\nabla_T \times \vec{\psi}_i) d\Omega + \int_{\Omega} (\nabla_T \times \vec{\psi}_j) \nabla_T \cdot (k_b \vec{\psi}_i) d\Omega \]

\[- \int_{\partial \Omega} (\nabla_T \cdot \vec{\psi}_j) (k_b \vec{\psi}_i) \cdot \hat{n} d\ell + \int_{\Omega} \vec{k}_e \vec{\psi}_j \cdot \vec{\psi}_i d\Omega \]
Reducing FEM Errors

• Error depends on match between true solution and basis functions

• To reduce error, can try the following:
  – H-adaptivity: decrease the mesh size
  – P-adaptivity: increase the degree of the fitted polynomials
  – HP-adaptivity: combine all of the above
Reducing FEM Errors

• Strategy for reducing errors:
  – Create an initial meshing
  – Compute solution on that meshing
  – Compute the error associated with it
  – If above our tolerance, refine the mesh spacing and start again
Next Class

• Is on Friday, Feb. 15
• Will continue with beam propagation method
• Recommended reading: Obayya, Sections 2.7-2.8