ECE 595, Section 10
Numerical Simulations
Lecture 29: Eigenmode Layered Computations (CAMFR)

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Recap from Friday: S-Matrices

• S-matrix method in plane wave basis
  – Calculate field eigenvectors in plane-wave basis
  – Calculate interface s-matrices and layer s-matrices
  – Compose S-matrices iteratively

• S4 simulation tool:
  – User interface
  – Lua commands
  – Results for example problems: multilayer stack; 1D grating; 2D Tikhodeev example
CAMFR: Rationale

• Many problems consist of layers with varying widths

• Examples:
  – LED stack
  – Rod-hole photonic crystal

• Natural form of solutions is semi-analytic, in terms of eigenmodes

CAMFR: Basic Strategy

• Break up structure into layers
• Calculate eigenmodes in each layer (of four types)
• Apply Lorentz reciprocity to match BC’s
• Propagate within layers using S-matrix method
• Apply inputs to calculate physical outputs
CAMFR: Eigenmode Decomposition

• This stage resembles BPM
• Begin with the Helmholtz equation:

\[ \nabla^2 + \epsilon \mu \omega^2 \psi = \beta^2 \psi \]

• Where \( \psi \) represents \( E \)-field or \( H \)-field, and \( \beta \) is the eigenvalue (wavevector along \( z \))
• Write 3D solutions in this form for each layer:

\[
\begin{pmatrix}
E(r) \\
H(r)
\end{pmatrix} = \sum_k A_k e^{-j\beta_k z} \begin{pmatrix}
E(r_t) \\
H(r_t)
\end{pmatrix}
\]
Can express eigenvalues in terms of $\text{Re} n_{\text{eff}}$ and $\text{Im} n_{\text{eff}}$
Eigenmode Classification

Guided mode

Im $\beta = 0$; discrete

Radiation mode

Re $\beta = 0$ or Im $\beta = 0$; continuous

Complex mode

Im $\beta \neq 0$; Re $\beta \neq 0$; discrete complex-conjugate pairs

Leaky mode

Im $\beta \neq 0$; Re $\beta \neq 0$; discrete

Lorentz Reciprocity

- Evaluate Maxwell’s equations across boundary using this surface

Lorentz Reciprocity

• Starting with Maxwell’s equations:

\[ \nabla \times E_1 = -j\omega \mu H_1 \quad \nabla \times E_2 = -j\omega \mu H_2 \]
\[ \nabla \times H_1 = J_1 + j\omega \varepsilon E_1 \quad \nabla \times H_2 = J_2 + j\omega \varepsilon E_2 \]

• Can form the expression:

\[ \nabla \cdot (E_1 \times H_2 - E_2 \times H_1) = J_1 \cdot E_2 - J_2 \cdot E_1 \]

• Integrating over \( V \) and using Gauss’ theorem:

\[ \int \int_S (E_1 \times H_2 - E_2 \times H_1) \cdot dS = \int \int \int_V (J_1 \cdot E_2 - J_2 \cdot E_1) dV \]
Lorentz Reciprocity

- Lorentz Reciprocity theorem becomes:

\[
\int \int_S \frac{\partial}{\partial z} (\mathbf{E}_1 \times \mathbf{H}_2 - \mathbf{E}_2 \times \mathbf{H}_1) \cdot \mathbf{u}_z dS = \int \int_S (\mathbf{J}_1 \cdot \mathbf{E}_2 - \mathbf{J}_2 \cdot \mathbf{E}_1) dS
\]

- For z-invariant media:

\[
\int \int_S (\mathbf{E}_{m,t} \times \mathbf{H}_{n,t}) \cdot \mathbf{u}_z dS = 0
\]

Boundary Conditions

• Assuming:

\[ E_{p,t}^I + \sum_j R_{j,p} E_{j,t}^I = \sum_j T_{j,p} E_{j,t}^{II} \]

• Defining overlap between modes to be:

\[ \langle E_m, H_n \rangle \equiv \int \int_S (E_m \times H_n) \cdot u_z dS \]

• We obtain the transmission coefficient:

\[ \sum_j [\langle E_i^I, H_j^{II} \rangle + \langle E_j^{II}, H_i^I \rangle] T_{j,p} = 2\delta_{ip} \langle E_p^I, H_p^I \rangle \]

• And reflection coefficient:

\[ R_{i,p} = \frac{1}{2 \langle E_i^I, H_i^I \rangle} \sum_j [\langle E_j^{II}, H_i^I \rangle - \langle E_i^I, H_j^{II} \rangle] T_{j,p} \]
S-Matrix Solution

• Now we can employ the standard S-matrix scheme from Li ’96:

\[
\begin{align*}
T_{1,p+1} &= t_{p,p+1} \cdot (I - R_{p,1} \cdot r_{p,p+1})^{-1} \cdot T_{1,p} \\
R_{p+1,1} &= t_{p,p+1} \cdot (I - R_{p,1} \cdot r_{p,p+1})^{-1} \cdot R_{p,1} \cdot t_{p+1,p} + r_{p+1,p} \\
R_{1,p+1} &= T_{p,1} \cdot (I - r_{p,p+1} \cdot R_{p,1})^{-1} \cdot r_{p,p+1} \cdot T_{1,p} + R_{1,p} \\
T_{p+1,1} &= T_{p,1} \cdot (I - r_{p,p+1} \cdot R_{p,1})^{-1} \cdot t_{p+1,p}
\end{align*}
\]

• We can compose the S-matrix starting from the identity matrix until we include all layers
Periodic Eigenproblems

• Periodic layered structures will:

\[
\begin{bmatrix}
A & B \\
C & D
\end{bmatrix}
\cdot
\begin{bmatrix}
F \\
B
\end{bmatrix}
= e^{-j k z p} \cdot
\begin{bmatrix}
F \\
B
\end{bmatrix}
\]

• Since T-matrix is nearly singular, use SVD:

\[
A = U \cdot \Sigma \cdot V^H
\]

• Where \( U \) and \( V \) are unitary; \( \Sigma \) diagonal. Then:

\[
A^{-1} = V \cdot \text{diag} \left( \frac{1}{\sigma_i} \right) \cdot U^H
\]
CAMFR: 2D Photonic Crystals

CAMFR: 2D PhC Waveguide

\[ f = \alpha / \lambda \]

Even mode
Next Class

• Is on Wednesday, March 27
• Will discuss CAMFR interface: http://camfr.sourceforge.net