ECE 595, Section 10
Numerical Simulations
Lecture 32: Simulations of Coupled Mode Theory (CMT)

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Recap from Friday

• Overview of Coupled Mode Theory
• Derivation of Coupled Mode Equations
• Applications:
  – Single Waveguides
  – Add-Drop filters
  – Waveguide Bends
  – Channel Drop
  – T-Splitters
  – Nonlinear Kerr Waveguides
Outline

• Recap from Friday
• Numerical ODE solvers
  – Initial value problems
  – Boundary value problems
• nanoHUB Tool – CMTcomb3:
  – Rationale
  – Governing ODEs
  – User interface
  – Output analysis
Numerical ODE Solvers

• Objective: to solve ODE (e.g., \( \frac{dX}{dt} = f(X) \)) with greatest accuracy and least computational cost

• Categories:
  – Initial value problems
  – Boundary value problems

• Algorithms:
  – Euler methods
  – Higher-order methods (e.g., Runge-Kutta)
  – Shooting methods
  – Finite element/difference methods
Numerical ODE IVP Solvers

• Euler Method: discretizes original ODE and solves in time steps of $\Delta t$:

$$\Delta X = \Delta t \cdot f(X)$$

• Advantages: fast, easy to implement

• Disadvantages: inaccurate for many ODEs with modest to large step sizes

• Problem is implicit linear evolution – fails badly for ODEs with significant curvature
Numerical ODE IVP Solvers

- Higher-order methods: incorporate more than just first derivative into solution
- In principle, incorporating nth derivative should reduce errors to $1/\Delta t^{n+1}$
- Example: leapfrog method
  - Let:
    \[
    \frac{d}{dt}(x) = \begin{pmatrix} v \\ F(x) \end{pmatrix}
    \]
  - Alternate updating $x$ and $v$:
    \[
    x_{i+1} = x_i + v_i \Delta t + \frac{1}{2} F(x_i) \Delta t^2 \\
    v_{i+1} = v_i + \frac{1}{2} [F(x_i) + F(x_{i+1})] \Delta t
    \]
**Numerical ODE IVP Solvers**

- Runge-Kutta methods: incorporate points with spacing less than $\Delta t$ for each higher accuracy (e.g., `ode45` in MATLAB)
- Generally define $k_i$ to incrementally determine slopes within interval:
  \[ k_i = \Delta t \cdot f(t_n + a_i \Delta t, y_n + \sum_{j=1}^{\infty} b_{ij} k_j) \]
- This gives rise to solutions:
  \[ y_{n+1} = y_n + \Delta t \sum_{i=1}^{M} c_i k_i \]
- In general, order of solution will be equal to $M$; accuracy will go as $1/\Delta t^{M+1}$
Numerical ODE Solvers

- Stiff solvers: for ODEs with a rapid harmonic oscillation, use backward differentiation formulae:
  \[
  \sum_{i=0}^{M} c_i y_{n+i} = \Delta t \cdot f(t_{n+M}, y_{n+M})
  \]
- Convergence can be orders of magnitude better
- Implemented with ode15s in MATLAB
Numerical ODE BVP Solvers

• Shooting method applies Euler method
• However, finite element and finite difference methods are ideal for boundary value problems
• Finite elements: discretize on finite element basis, and solve using Galerkin method
• Finite difference: discretize on grid, and solve using leapfrog method
CMTComb3 – a nanoHUB.org tool

Goal is to reduce physics of four-wave mixing in a microring resonator to a set of coupled mode equations, then solve them.

Recap: Derivation of CMT Equations

- By linearity, coupling of waveguides into modes given by:
  \[ \frac{dA_i}{dt} = \ldots + \sum_j \alpha_{ij} S_{j+} \]

- For similar reasons, outgoing waveguide modes given by:
  \[ S_{i-} = \beta_i S_{i+} + \sum_j \gamma_{ij} A_j \]

- By conservation of energy, inputs must be stored or lost:
  \[ \sum_i \left[ |S_{i+}|^2 - |S_{i-}|^2 - \frac{dU_i}{dt} \right] = 0 \]

- Special cases can be used to obtain coefficients: \( \{\alpha_{ij}, \beta_i, \gamma_{ij}\} \)
CMTComb3 – a nanoHUB.org tool

• Basic equations

Pump mode:
\[ \frac{d a_k}{d t} = (i \omega_k - \Gamma_k - \gamma_k) a_k + i \sum_{l,m}^N k_{lmk} a_l^* a_m a_{k+l-m} + \sqrt{2 \Gamma_k} s_+ \]

Side modes:
\[ \frac{d a_k}{d t} = (i \omega_k - \Gamma_k - \gamma_k) a_k + i \sum_{l,m}^N k_{lmk} a_l^* a_m a_{k+l-m} \]

Output mode:
\[ \frac{d s_-}{d t} = - \frac{d s_+}{d t} + \sum_{k=1}^N \sqrt{2 \Gamma_k} \frac{d a_k}{d t} \]

CMTComb3 – a nanoHUB.org tool

Input parameters for user to modify:
- Number of modes
- Plot modes
- Pump frequency
- Input scale factor
- Loss rate $\Gamma$
- Nonlinear strength $\kappa$
CMTComb3 – a nanoHUB.org tool
CMTComb3 – a nanoHUB.org tool

- Coupled mode theory (CMT) is accurate in the weak-coupling regime, and much faster than full-wave time-domain simulations of comparable systems

- Our CMT comb simulation enables exploration of basic physics and rapid prototyping for specific applications, including metrology and RF signal modulation
Next Class

• Is on Wednesday, April 3
• Next time: we will discuss finite-difference time domain techniques
• Suggested reference: S. Obayya’s book, Chapter 5, Sections 4-6