ECE695: Reliability Physics of Nano-Transistors
Lecture 32: Physical vs. Empirical distribution

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Outline

1. Physical Vs. empirical distribution
2. Properties of classical distribution function
3. Moment-based fitting of data
4. Conclusions
Data vs. Hypothesis

Outliers identified (box-plot, Chauvenet)

Trend identified using median based plotting, stem-leaf histogram

CDF plotted using Kaplan-Meier formula

Non-parametric bootstrap to identify parameter uncertainty

Empirical reliability (Hypothesis testing)

Statistical reliability (Series/parallel systems)

Physical Reliability (Distribution function, prediction of an analytical model)
Statistical Distribution is Physical

Experiments

If a problem can be mapped into one of the well known family, large number of results are available.
Outline

1. Physical Vs. empirical distribution
2. Parametric Vs. non-parametric fits
3. Estimating various distribution functions
4. Conclusions
Choosing distribution function

People choose functions that describe wide range of phenomena

- **Normal**: After all, everything eventually becomes normal (not really!) Distribution of last resort.

- **Log-Normal**: A variant of normal distribution that seems to describe many reliability problems phenomenologically (correlated processes, such as electromigration in interconnects, shunt distribution in solar cells)

- **Weibull**: Many physical systems are described by it. In the limiting case, it becomes Exponential distribution (extreme value problems such as thin oxide breakdown)
Two parameter family: Normal distribution

\[ f(t; \mu, \sigma) = \frac{1}{\sigma \sqrt{2\pi}} \cdot \exp \left[ -\frac{(t - \mu)^2}{2\sigma^2} \right] \]

\[ F(t) = \Phi \left( \sigma^{-1}(t - \mu) \right) \]

\[ \Phi(z) = \left[ 1 + \text{erf}(z/\sqrt{2}) \right] / 2 \]

\( \mu \) = average, \( \sigma \) = standard deviation

Binomial distribution, Poisson distribution, chi-square, student-t distribution …

MATpd = fitdist(data,'Normal')

http://en.wikipedia.org/wiki/Normal_distribution
Two parameter family: log-normal distribution

(PDF) $f(t; \mu, \sigma) = \frac{1}{t \times \sigma \sqrt{2\pi}} \cdot \exp\left[-\frac{(\ln(t) - \ln(\mu))^2}{2\sigma^2}\right]$

$\mu=$average, $\sigma=$standard deviation

(CDF) $F(t) = \Phi\left(\sigma^{-1} \ln\frac{t}{\mu}\right)$  \hspace{1cm} $\Phi(z) = \left[1 + \text{erf}\left(z/\sqrt{2}\right)\right]/2$

$\sigma = \ln(t_2/t_1) / \left[\Phi^{-1}(F(t_2)) - \Phi^{-1}(F(t_1))\right]$

$= \ln(t_2 \at F = 0.5 / t_1 \at F = 0.159)$

$\lambda(t) = \sqrt{\frac{2}{\pi}} \frac{1}{t\sigma} \exp\left[-\sigma^2 \left\{\ln(t/\mu)\right\}^2 / 2\right]$

http://en.wikipedia.org/wiki/Log-normal_distribution
Two parameter family: Weibull distribution

\[ f(t; \alpha, \beta) = \frac{\beta}{\alpha^\beta} \cdot t^{\beta-1} \cdot e^{-(t/\alpha)^\beta} \quad (\alpha, \beta > 0) \]

\[ F(t) = 1 - \exp\left(-\left(t/\alpha\right)^\beta\right) \]

\[ \lambda(t) = \frac{\beta t^{\beta-1}}{\alpha^\beta} \]

\( \beta \) = shape parameter, \( \alpha \) = scale parameter

\( \beta = 1 \) …. Exponential distribution

Memory-less distribution, constant hazard rate

\( \beta = 2 \) …. Rayleigh distribution

Light scattering, Corrosion in contacts, Failure rate increases with time

http://en.wikipedia.org/wiki/Weibull_distribution
# Empirical statistical distributions

<table>
<thead>
<tr>
<th>Distribution</th>
<th>Normal</th>
<th>Log Normal</th>
<th>Weibull</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>PDF</strong></td>
<td>$f(t) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(t-\mu)^2}{2\sigma^2}}$</td>
<td>$f(t) = \frac{1}{t\sigma\sqrt{2\pi}} e^{-\frac{(\ln t - \ln \mu)^2}{2\sigma^2}}$</td>
<td>$f(t) = \frac{\beta}{\alpha} \left(\frac{t}{\alpha}\right)^{\beta-1} \cdot e^{-\left(\frac{t}{\alpha}\right)^\beta}$</td>
</tr>
<tr>
<td><strong>PDF</strong></td>
<td><img src="image1" alt="Normal PDF" /></td>
<td><img src="image2" alt="Log Normal PDF" /></td>
<td><img src="image3" alt="Weibull PDF" /></td>
</tr>
<tr>
<td><strong>CDF</strong></td>
<td>$F(t) = \frac{1}{2} + \frac{1}{2} \text{erf} \left( \frac{t - \mu}{\sigma\sqrt{2}} \right)$</td>
<td>$F(t) = \frac{1}{2} + \frac{1}{2} \text{erf} \left( \frac{\ln t - \ln \mu}{\sigma\sqrt{2}} \right)$</td>
<td>$1 - e^{-\left(\frac{t}{\alpha}\right)^\beta}$</td>
</tr>
<tr>
<td><strong>CDF</strong></td>
<td><img src="image4" alt="Normal CDF" /></td>
<td><img src="image5" alt="Log Normal CDF" /></td>
<td><img src="image6" alt="Weibull CDF" /></td>
</tr>
</tbody>
</table>

Asymmetric Weibull distributions for $\beta = 1$ and $\beta = 3$.
# Empirical statistical distributions

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</tr>
<tr>
<td><strong>CDF</strong></td>
<td>[F(t)]</td>
<td>[F(t)]</td>
<td>[\ln(-\ln(1-F(t)))]</td>
</tr>
<tr>
<td><strong>Moment 1st</strong></td>
<td>[\mu]</td>
<td>[\mu \cdot e^{\frac{\sigma^2}{2}}]</td>
<td>[\alpha \sqrt{\frac{1}{\beta}}]</td>
</tr>
<tr>
<td><strong>Moment 2nd</strong></td>
<td>[\sigma^2]</td>
<td>[2\mu \cdot (e^{\sigma^2} - 1) \cdot e^{\sigma^2}]</td>
<td>[\sqrt{\frac{\alpha^2}{\beta} - \alpha^2 \Gamma^2 (1 + \frac{1}{\beta})}]</td>
</tr>
</tbody>
</table>
### Definitions of distribution functions

<table>
<thead>
<tr>
<th>Name</th>
<th>Symbol</th>
<th>Expression</th>
</tr>
</thead>
<tbody>
<tr>
<td>Prob. distribution</td>
<td>$f(t;\alpha,\beta,...)$</td>
<td>$f(t;\alpha,\beta,...)$</td>
</tr>
<tr>
<td>cumulative PDF</td>
<td>$F(t)$</td>
<td>$\int_{-\infty}^{t} f(t')dt'$</td>
</tr>
<tr>
<td>survival function</td>
<td>$R(t)$</td>
<td>$1 - F(t)$</td>
</tr>
<tr>
<td>hazard rate</td>
<td>$\lambda(t)$</td>
<td>$\frac{f(t)}{1 - F(t)}$</td>
</tr>
<tr>
<td>cum. hazard rate</td>
<td>$H(t)$</td>
<td>$\int_{0}^{t} \lambda(t')dt'$</td>
</tr>
<tr>
<td>average hazard</td>
<td>$\lambda_c(t)$</td>
<td>$\frac{1}{t} \int_{0}^{t} \lambda(t')dt'$</td>
</tr>
</tbody>
</table>

These functions are used in different fields in different ways …
Example: Derivation of hazard function

\( H(t) \) … Probability that having survived till time \( t \) (event \( A \)), it fails within time \( (t+dt) \) (event \( B \))

\[
P(B \mid A) = \frac{P(A \ast B)}{P(A)} = \frac{P(B)}{P(A)} = \frac{f(t)dt}{R(t)} \Rightarrow h(t) = \frac{f(t)}{R(t)}
\]

B includes A …. (failed before)

Integrated Hazard till time \( t \) ….

\[
H(t) = \int_{0}^{t} h(u)du = -\ln R(t)
\]

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Discrete Transform: because data is discrete

\[ F_i = \frac{i - \alpha}{n - 2\alpha + 1} \]

\[ f_i = \frac{dF_i}{dt} = \frac{F_{i+1} - F_i}{t_{i+1} - t_i} = \frac{1}{(n - 2\alpha + 1)(t_{i+1} - t_i)} \]

\[ \lambda_i = \frac{f_i}{1 - F_i} = \frac{1}{(n - i - \alpha + 1)(t_{i+1} - t_i)} \]
<table>
<thead>
<tr>
<th></th>
<th>(f(t))</th>
<th>(F(t))</th>
<th>(R(t))</th>
<th>(\lambda(t))</th>
<th>(H(t))</th>
<th>(\lambda_c(t))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(f(t))</td>
<td>(f(t))</td>
<td>(\frac{dF(t)}{dt})</td>
<td>(-\frac{dR(t)}{dt})</td>
<td>(\lambda(t)e^{-\int_0^t \lambda(t')dt'})</td>
<td>(e^{-H(t)})</td>
<td>((\lambda_c + t \frac{d\lambda_c(t)}{dt})e^{-\lambda_c(t)t})</td>
</tr>
<tr>
<td>(F(t))</td>
<td>(\int_0^t f(t')dt')</td>
<td>(F(t))</td>
<td>(1 - R(t))</td>
<td>(1 - e^{-\int_0^t \lambda(t')dt'})</td>
<td>(1 - e^{-H(t)})</td>
<td>(1 - e^{-\lambda_c(t)t})</td>
</tr>
<tr>
<td>(R(t))</td>
<td>(\int_t^\infty f(t')dt')</td>
<td>(1 - F(t))</td>
<td>(R(t))</td>
<td>(e^{-\int_0^t \lambda(t')dt'})</td>
<td>(e^{-H(t)})</td>
<td>(e^{-\lambda_c(t)t})</td>
</tr>
<tr>
<td>(\lambda(t))</td>
<td>(\frac{f(t)}{\int_t^\infty f(t')dt'})</td>
<td>(-\frac{d \ln(1 - F(t))}{dt})</td>
<td>(-\frac{d \ln R(t)}{dt})</td>
<td>(\lambda(t))</td>
<td>(\frac{dH(t)}{dt})</td>
<td>(\lambda_c + t \frac{d\lambda_c(t)}{dt})</td>
</tr>
<tr>
<td>(H(t))</td>
<td>(-\ln\left(\int_t^\infty f(t')dt'\right))</td>
<td>(-\ln(1 - F(t)))</td>
<td>(-\ln R(t))</td>
<td>(\int_0^t \lambda(t')dt')</td>
<td>(H(t))</td>
<td>(t\lambda_c(t))</td>
</tr>
<tr>
<td>(\lambda_c(t))</td>
<td>(-\frac{1}{t}\ln\left(\int_t^\infty f(t')dt'\right))</td>
<td>(-\frac{\ln(1 - F(t))}{t})</td>
<td>(-\frac{\ln R(t)}{t})</td>
<td>(1/t\int_0^t \lambda(t')dt')</td>
<td>(\frac{H(t)}{t})</td>
<td>(\lambda_c(t))</td>
</tr>
</tbody>
</table>

**HW:** Derive a few reliability functions yourself ...
Series and parallel systems: how to use the distribution functions

1 - \(F_s(t) = [1 - F_1(t)] \times [1 - F_2(t)] \times ...\)

\(R_s(t) = R_1(t) \times R_2(t) \times ...\)

\[\lambda = \frac{dF}{dt} = \frac{-dR}{dt} = -\frac{d}{dt} \ln R\]

\(\lambda_s(t) = \lambda_1(t) + \lambda_2(t) + ...\)

\(F_s(t) = F_1(t) \times F_2(t) \times ...\)

1 - \(R_s(t) = [1 - R_1(t)] \times [1 - R_2(t)] \times ...\)

Advantage of redundant system:

\[\frac{\lambda_i(t)}{\lambda_s(t)} = 1 + \frac{F + F^2 + ... F^{n-1}}{nF^{n-1}}\]

R1, R2, ... may have different distributions
Outline

1. Physical Vs. empirical distribution
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Matching moments to distributions

Of 60 oxides, 7 failed in 1000 hrs

<table>
<thead>
<tr>
<th>Rank</th>
<th>Lifetime</th>
<th>( F_i = (i-0.3)/(n+0.4) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>181</td>
<td>0.012</td>
</tr>
<tr>
<td>2</td>
<td>299</td>
<td>0.028</td>
</tr>
<tr>
<td>3</td>
<td>389</td>
<td>0.045</td>
</tr>
<tr>
<td>4</td>
<td>430</td>
<td>0.061</td>
</tr>
<tr>
<td>5</td>
<td>535</td>
<td>0.078</td>
</tr>
<tr>
<td>6</td>
<td>610</td>
<td>0.094</td>
</tr>
<tr>
<td>7</td>
<td>805</td>
<td>0.111</td>
</tr>
</tbody>
</table>

Log-Normal Distribution Parameters
\( s = \ln(T_{50%}/T_{15.9%}) \), \( \sigma = \ln(3600/980) = 1.30 \)
\( \mu = \ln(T_{50%}) = \ln(3600) = 8.19 \)

Weibull Distribution Parameters
When \( t = \alpha \), \( \ln(1-F(t)) = -1 \), \( F(t) = 0.632 \), \( \alpha = 2990 \)
\( \beta \) estimated using parameter fitting as 1.56
Problem of matching the moments

Log-normal distribution is considerably optimistic
Problem of matching distribution: BFRW

\[ \frac{\partial P}{\partial t} = D \frac{\partial^2 P}{\partial x^2} \]
\[ P(x, t = 0) = \delta(x - x_0) \]
\[ P(x = 0, t) = 0 \]

\[ P(x, t) = (4\pi Dt)^{-1/2} \left[ e^{-(x - x_0)^2/4Dt} - e^{-(x + x_0)^2/4Dt} \right] \]

\[ \int_0^t f(\tau)d\tau + \int_0^L P(x, t)dx = 1 \Rightarrow f(t) = \frac{x_0}{\sqrt{4\pi Dt^3}} e^{-x_0^2/4Dt} \]
Match apparently reasonable, but wrong

\[ f_G(t) = \frac{t^{k-1}e^{-t/\theta}}{\Gamma(k)\theta^k} \quad T_{avg} = k\theta \]

\[ PDF = \frac{x_0}{\sqrt{4\pi Dt}} e^{-x_0^2/4Dt} \rightarrow t^{-3/2} \]

\[ \text{Monte Carlo} \]

1000, 10000, 50000

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Conclusions

1. Once the data is plotted using the principles discussed in lecture, one should develop a statistical model for the phenomena involved.

2. If unsuccessful, one should choose functions with least number of variables that described the system. In reliability, 2-parameter family such as log-normal, Weibull distributions are most popular, because they have been seen to give good results for correlated and weakest-link problems.

3. Reliability is an extreme value problem, therefore one should pay particular attention to tails of the distribution and choose sample size accordingly.

4. Moment-based methods are popular, but cannot distinguish between the tails of the distribution (reflects very high moments).
D. C. Hoaglen, F. Mosteller, and J.W. Tukey, “Understanding Robust and Exploratory Data Analysis”, Wiley Interscience, 1983. Explains the importance of Median based analysis when the dataset is small and the quality cannot be guaranteed.


AT&T, “Statistical Quality Control Handbook”.


