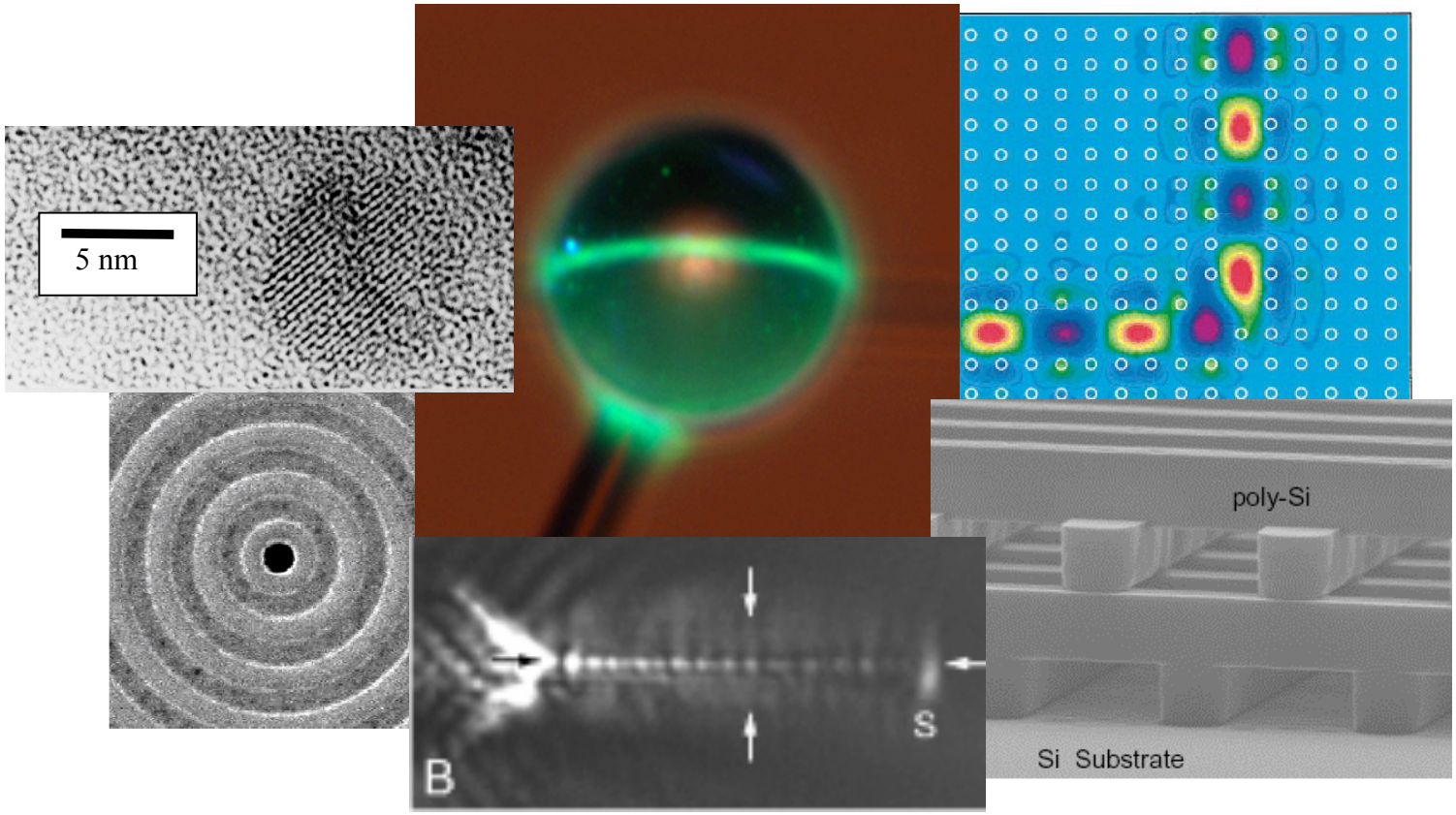
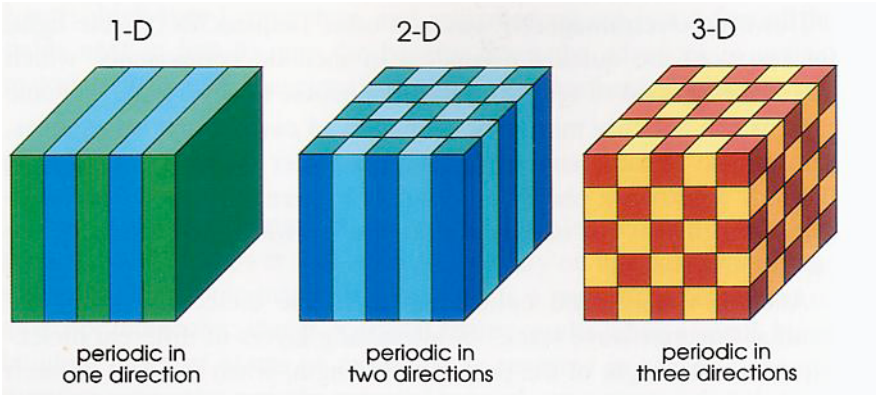
Lecture 5: Photonic Crystals – Introduction



Photonic crystal: An Introduction



Photonic crystal:

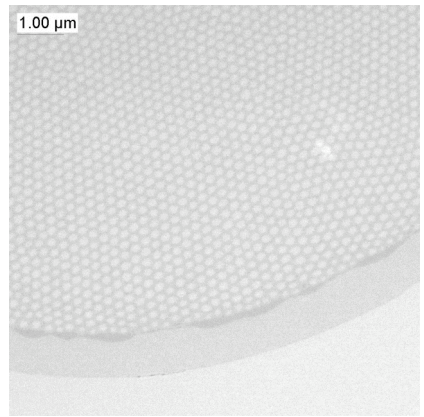
Periodic arrangement of dielectric (metallic, polaritonic…) objects.
Lattice constants comparable to the wavelength of light in the material.

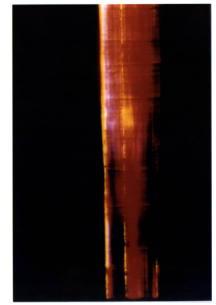
“ A worm ahead of its time”

Sea Mouse

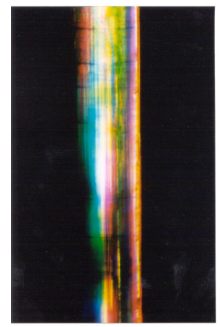
20cm

and its hair





Normal incident light



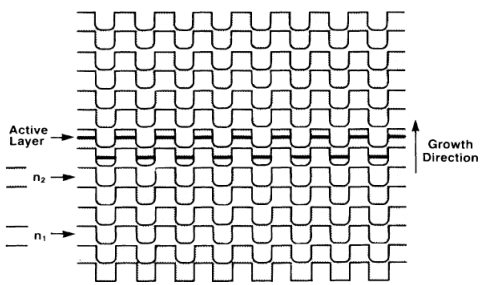
Off-Normal incident light

http://www.physics.usyd.edu.au/~nicolae/seamouse.html

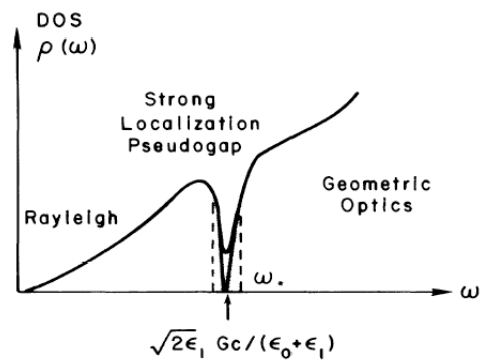
Fast forward to 1987……

E. Yablonovitch
"Inhibited spontaneous emission in solid state physics and electronics"
Physical Review Letters, vol. 58, pp. 2059, 1987

S. John
"Strong localization of photons in certain disordered dielectric superlattices"
Physical Review Letters, vol. 58, pp. 2486, 1987

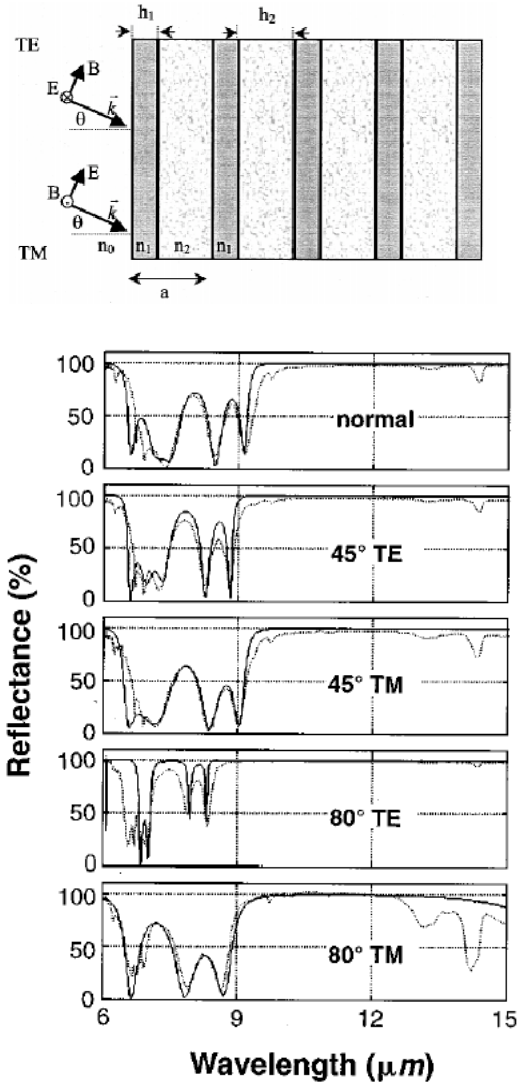


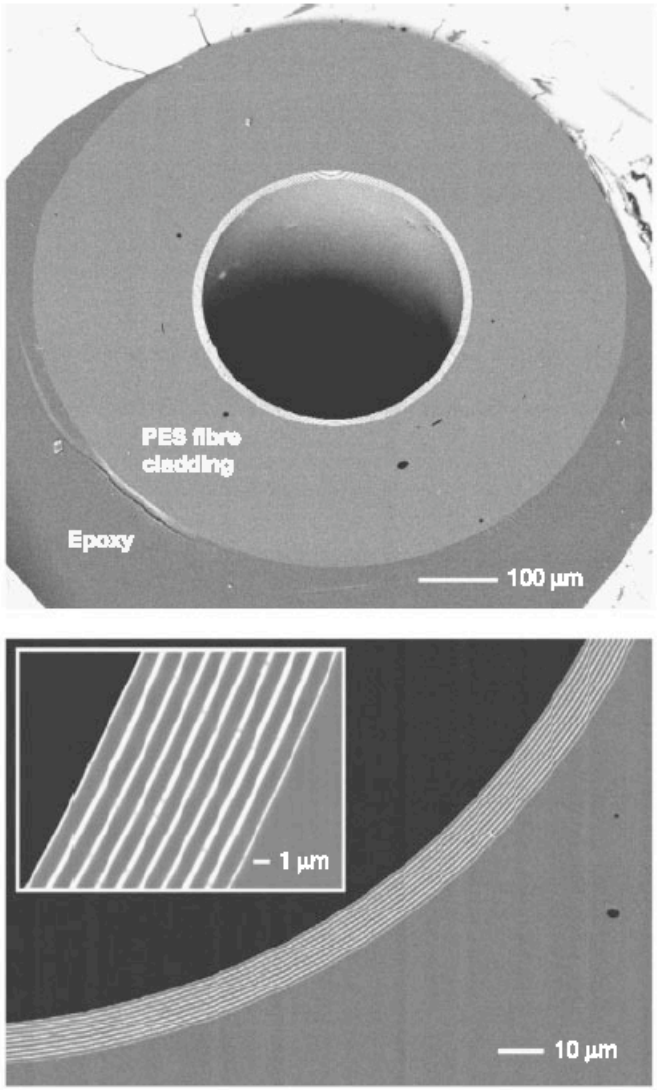
Face-centered cubic lattice



Complete photonic band gap

Omni-directional reflector

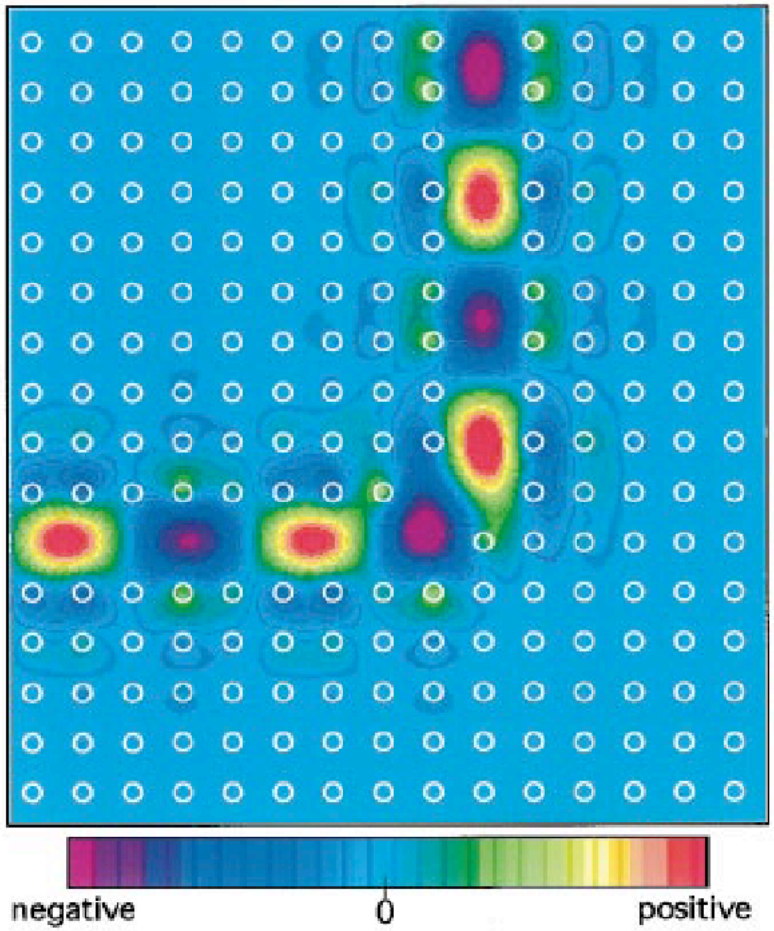


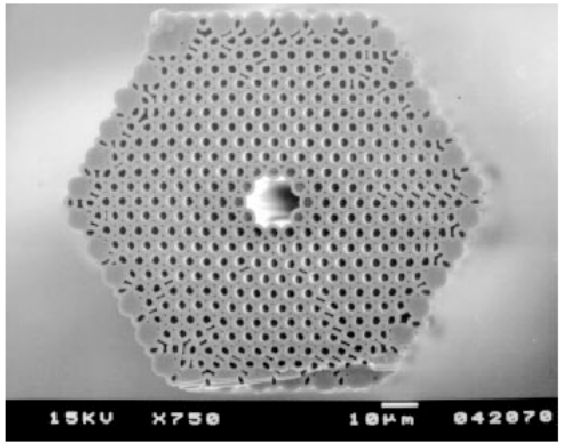


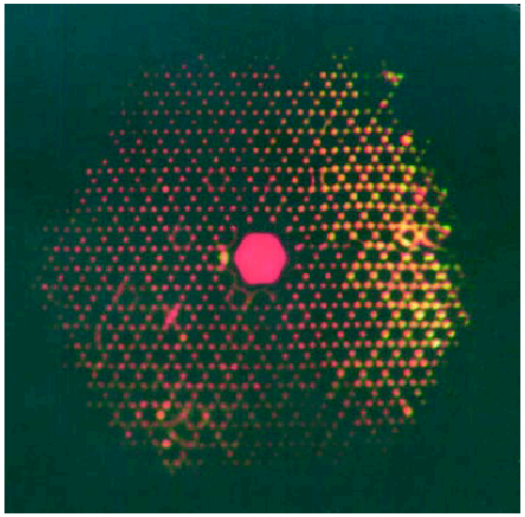
Y. Fink, et al, Science, vol.282, p.1679 (1998)

B. Temelkuran et al, Nature, vol.420, p.650 -3 (2002)

Integrated photonic circuits and photonic crystal fibers



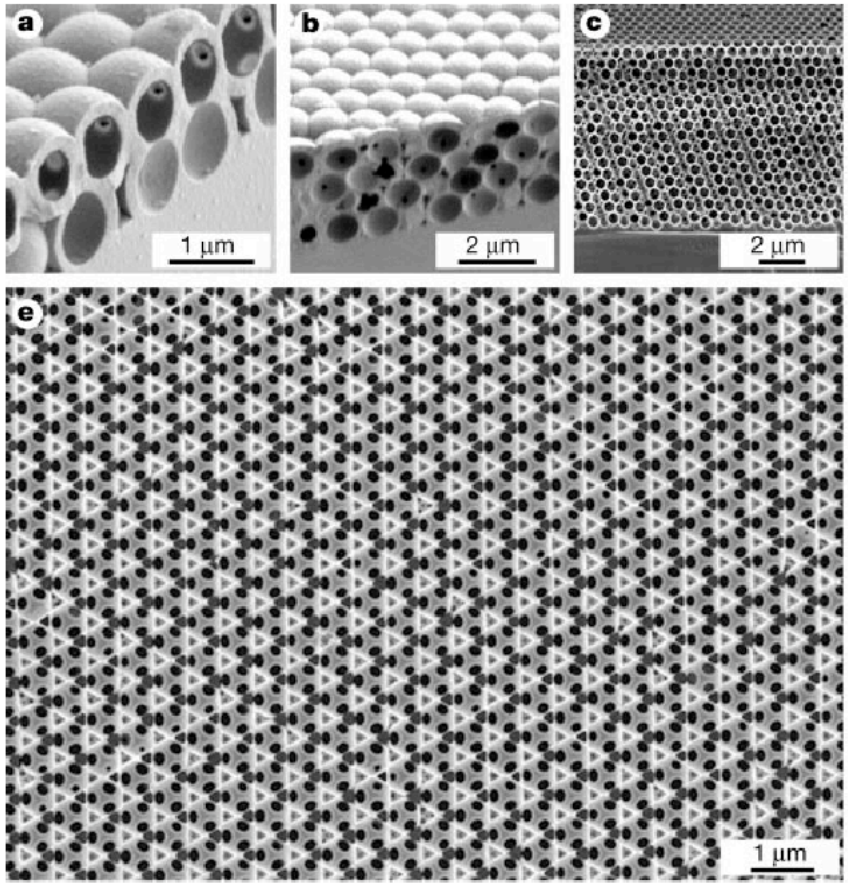




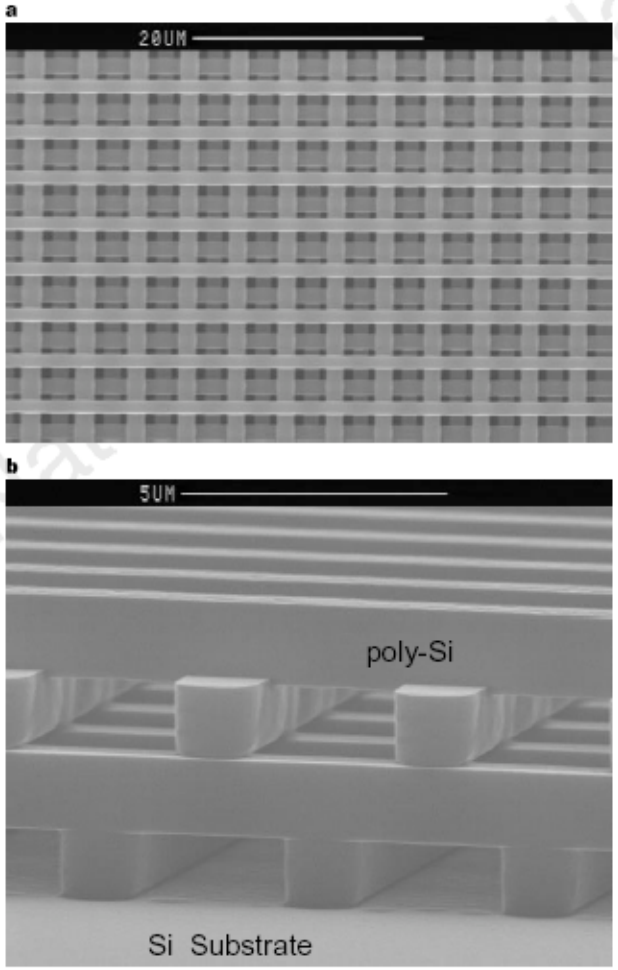
J. D. Joannopoulos, et al, Nature, vol.386, p.143-9 (1997)

R. F. Cregan, et al, Science, vol.285, p.1537-9 (1999)

Three-dimensional photonic crystals



Y. A. Vaslov, Nature, vol.414, p.289-93 (2001)



S. Lin et al, Nature, vol. 394, p. 251-3, (1998)

The emphasis of recent breakthroughs

- The use of strong index contrast, and the developments of nano-fabrication technologies, which leads to entirely new sets of phenomena.

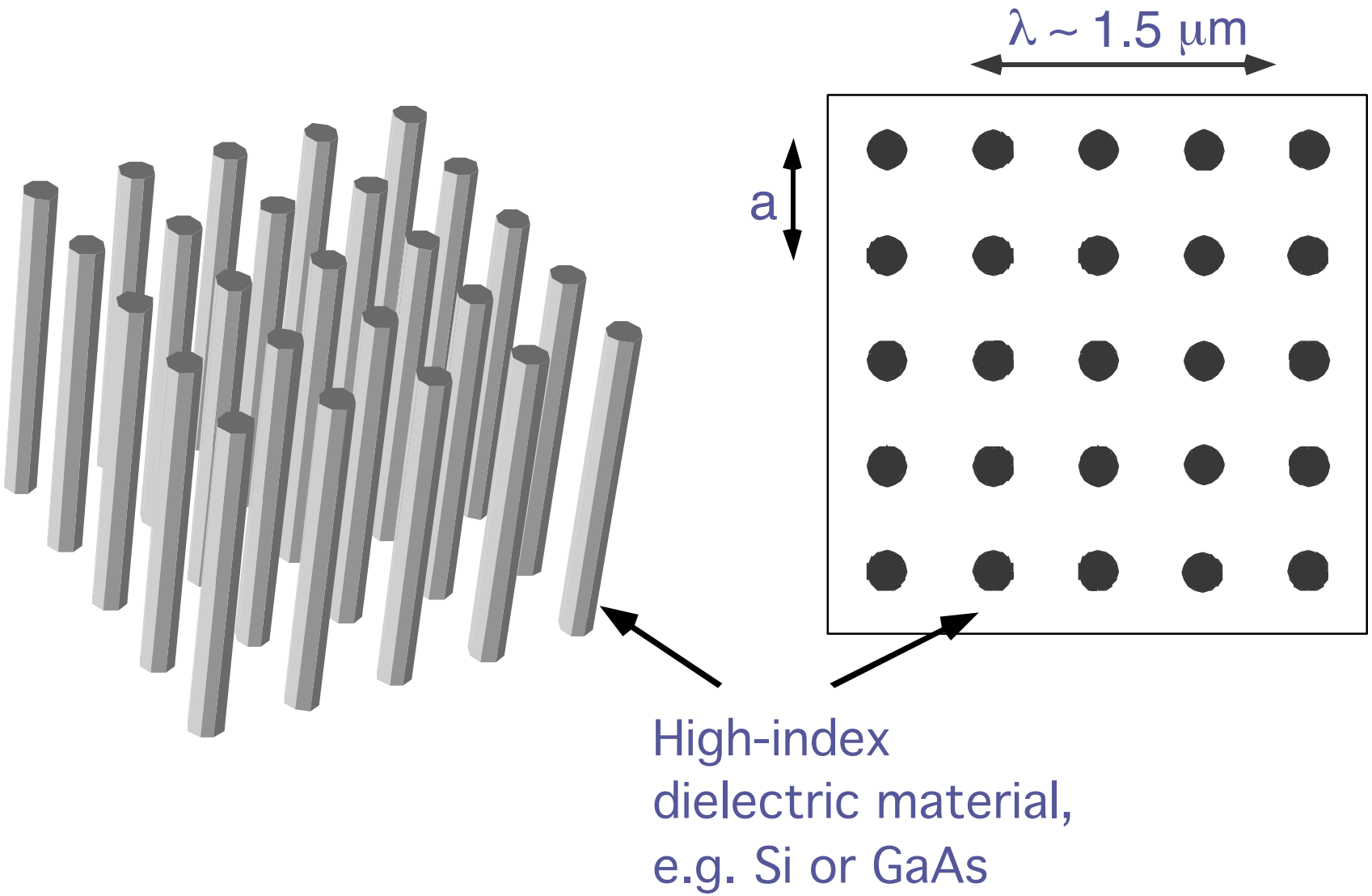
  Conventional silica fiber, $\delta n \sim 0.01$, photonic crystal structure, $\delta n \sim 1$

- New conceptual framework in optics

  Band structure concepts.
  Coupled mode theory approach for photon transport.

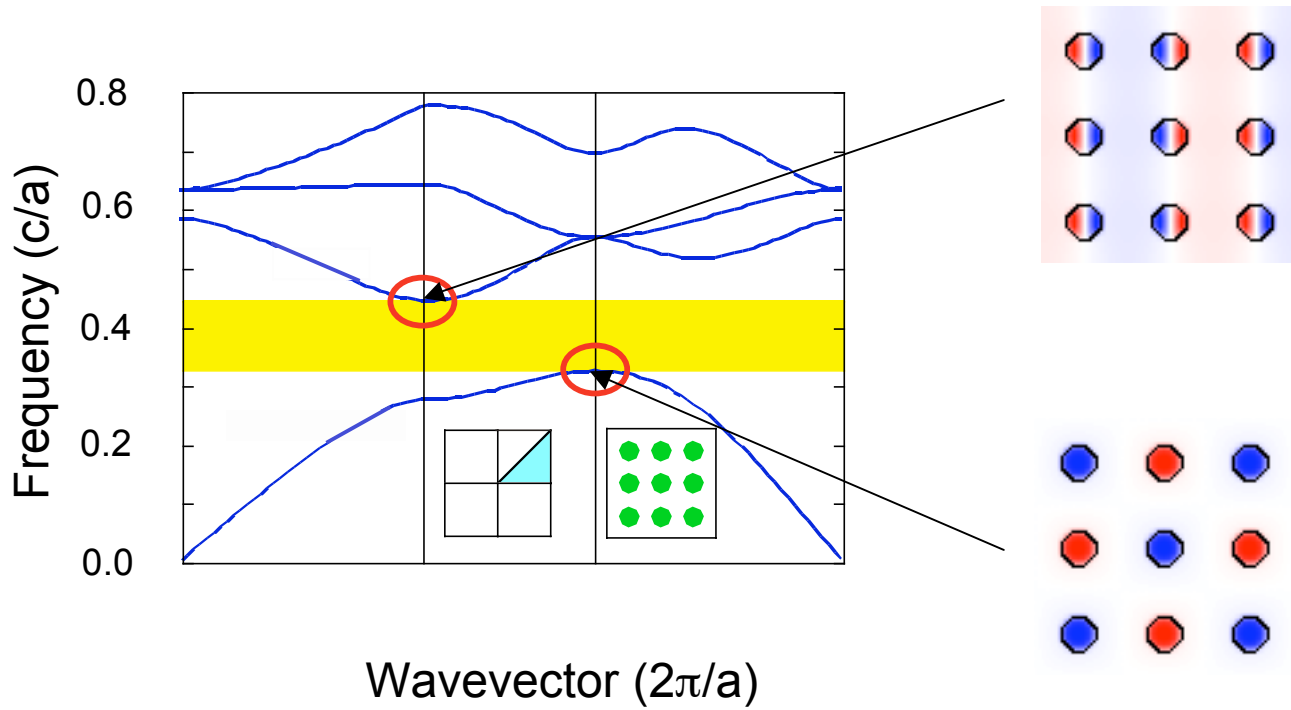
- Photonic crystal: semiconductors for light.

Two-dimensional photonic crystal



Band structure of a two-dimensional crystal

Displacement field parallel to the cylinder



Wavevector determines the phase between nearest neighbor unit cells.

X: (0.5*2π/a, 0): Thus, nearest neighbor unit cell along the x-direction is 180 degree out-of-phase

M: (0.5*2π/a, 0.5*2π/a): nearest neighbor unit cell along the diagonal direction is 180 degree out-of-phase

Maxwell's equation in the steady state

Time-dependent Maxwell's equation in dielectric media:

$$\nabla \bullet \mathbf{H}(\mathbf{r},t) = 0 \qquad \nabla \times \mathbf{H}(\mathbf{r},t) - \varepsilon(\mathbf{r}) \frac{\partial(\varepsilon_0 \mathbf{E}(\mathbf{r},t))}{\partial \mathbf{t}} = 0$$

$$\nabla \bullet \varepsilon \mathbf{E}(\mathbf{r},t) = 0 \qquad \nabla \times \mathbf{E}(\mathbf{r},t) + \frac{\partial(\mu_0 \mathbf{H}(\mathbf{r},t))}{\partial \mathbf{t}} = 0$$

Time harmonic mode (i.e. steady state):

$$\mathbf{H}(\mathbf{r},t) = \mathbf{H}(\mathbf{r}) e^{-i\omega t}$$

$$\mathbf{E}(\mathbf{r},t) = \mathbf{E}(\mathbf{r}) e^{-i\omega t}$$

Maxwell equation for the steady state:

$$\nabla \times \mathbf{H}(\mathbf{r}) + i\omega(\varepsilon(\mathbf{r})\varepsilon_0 \mathbf{E}(\mathbf{r})) = 0$$

$$\nabla \times \mathbf{E}(\mathbf{r}) - i\omega(\mu_0 \mathbf{H}(\mathbf{r})) = 0$$

Master's equation for steady state in dielectric

Expressing the equation in magnetic field only:

$$\nabla \times \frac{\mathbf{1}}{\varepsilon(\mathbf{r})} \nabla \times \mathbf{H}(\mathbf{r}) = \left(\frac{\omega}{c}\right)^2 \mathbf{H}(\mathbf{r})$$

$$c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}$$

Thus, the Maxwell's equation for the steady state can be expressed in terms of an eigenvalue problem, in direct analogy to quantum mechanics that governs the properties of electrons.

	Quantum mechanics	Electromagnetism
Field	$\Psi(\mathbf{r},t) = \Psi(\mathbf{r})e^{j\omega t}$	$\mathbf{H}(\mathbf{r},t) = \mathbf{H}(\mathbf{r})e^{i\omega t}$
Eigen-value problem	$\hat{H}\Psi(\mathbf{r}) = E\Psi(\mathbf{r})$	$\Theta\mathbf{H}(\mathbf{r}) = \left(\frac{\omega^2}{c^2}\right)\mathbf{H}(\mathbf{r})$
Operator	$\hat{H} = \frac{-\hbar^2\nabla^2}{2m} + V(\mathbf{r})$	$\Theta = \nabla \times \frac{\mathbf{1}}{\varepsilon(\mathbf{r})} \nabla \times$

Electromagnetism as an eigenvalue problem

The master equations define an operator:

$$\Theta\mathbf{H}(\mathbf{r}) \equiv \nabla \times \frac{\mathbf{1}}{\varepsilon(\mathbf{r})} \nabla \times \mathbf{H}(\mathbf{r})$$

Importantly, the Θ operator is a Hermitian operator. If we define the inner product of two vector fields **F**(r) and **G**(r) as:

$$(\mathbf{F},\mathbf{G}) = \int d\mathbf{r}\mathbf{F}^*(\mathbf{r}) \cdot \mathbf{G}(\mathbf{r})$$

then

$$\begin{aligned}
(\mathrm{F},\Theta\mathrm{G}) &= \int d\mathrm{r}\mathrm{F}^* \cdot \nabla \times \left(\frac{1}{\varepsilon} \nabla \times \mathrm{G} \right) \\
&= \int d\mathrm{r} (\nabla \times \mathrm{F})^* \cdot \left(\frac{1}{\varepsilon} \nabla \times \mathrm{G} \right) \\
&= \int d\mathrm{r} \left(\frac{1}{\varepsilon} \nabla \times \mathrm{F} \right)^* \cdot (\nabla \times \mathrm{G}) \\
&= \int d\mathrm{r} \left(\nabla \times \frac{1}{\varepsilon} \nabla \times \mathrm{F} \right)^* \cdot \mathrm{G} = (\Theta\mathrm{F},\mathrm{G})
\end{aligned}$$

General property of the harmonic modes

Having the Θ operator to be Hermitian leads to a number of nice properties about the harmonic modes

Assuming that **H**(r) is an eigen-mode, i.e. $\Theta\mathbf{H}(\mathbf{r}) = \left(\frac{\omega}{c}\right)^2 \mathbf{H}(\mathbf{r})$

ω^2 is real.

$$(\mathbf{H},\Theta\mathbf{H}) = \left(\frac{\omega}{c}\right)^2 (\mathbf{H},\mathbf{H}) = (\Theta\mathbf{H},\mathbf{H}) = \left(\frac{\omega^*}{c}\right)^2 (\mathbf{H},\mathbf{H})$$

ω^2 is positive.

$$\left(\frac{\omega}{c}\right)^2 (\mathbf{H},\mathbf{H}) = (\mathbf{H},\Theta\mathbf{H}) = \int d\mathbf{r} \frac{1}{\varepsilon(\mathbf{r})} |\nabla \times \mathbf{H}|^2$$

Two modes $\mathbf{H}_1$(r) and $\mathbf{H}_2$(r) at different frequencies ω_1 and ω_2 are orthogonal, i.e. $(\mathbf{H}_1, \mathbf{H}_2) = 0$

$$\left(\frac{\omega_1}{c}\right)^2 (\mathbf{H}_2,\mathbf{H}_1) = (\mathbf{H}_2,\Theta\mathbf{H}_1) = (\Theta\mathbf{H}_2,\mathbf{H}_1) = \left(\frac{\omega_2}{c}\right)^2 (\mathbf{H}_2,\mathbf{H}_1)$$

Thus if ω_1 and ω_2 are different, then $(H_1, H_2) = 0$.

Consequence of the orthogonality

For two real one-dimensional function *f*(x) and *g*(x) to be orthogonal, i.e.

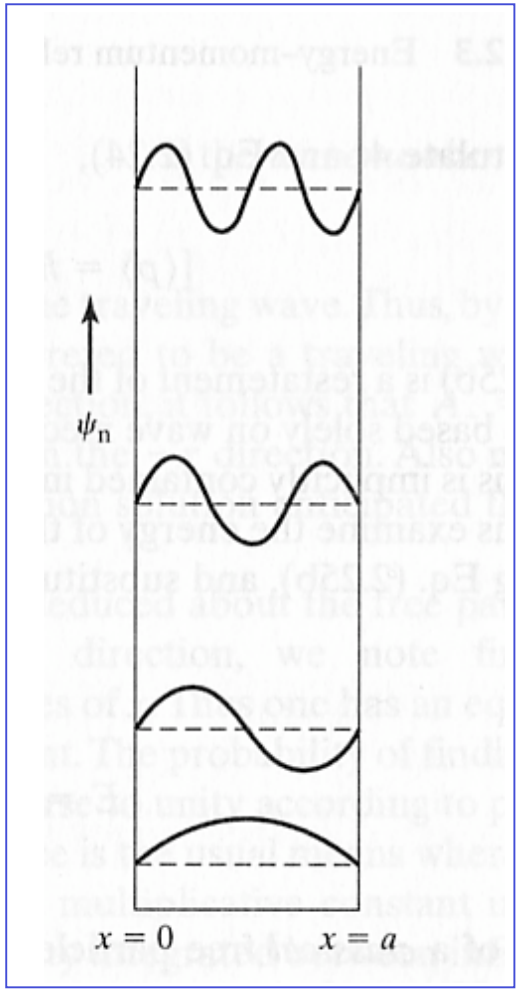
$$0 = (f, g) = \int dx f(x) g(x)$$

Thus, the product *fg* must be negative as much as it is positive over the interval of interest, so that the net integral vanishes.

Since the operator $\Theta \mathbf{H}(\mathbf{r}) \equiv \nabla \times \frac{1}{\varepsilon(\mathbf{r})} \nabla \times \mathbf{H}(\mathbf{r})$

contains derivative with respect to the field, higher-frequency mode tends to have more spatial variation in their field patterns.

By orthogonality, higher-frequency mode tends to have more nodal plane in the field pattern.



Scale Invariance

The solution at one scale determines the solution at all other length scales.

Suppose, for example, we have an electromagnetic steady state H(r) in a dielectric configuration ε(r)

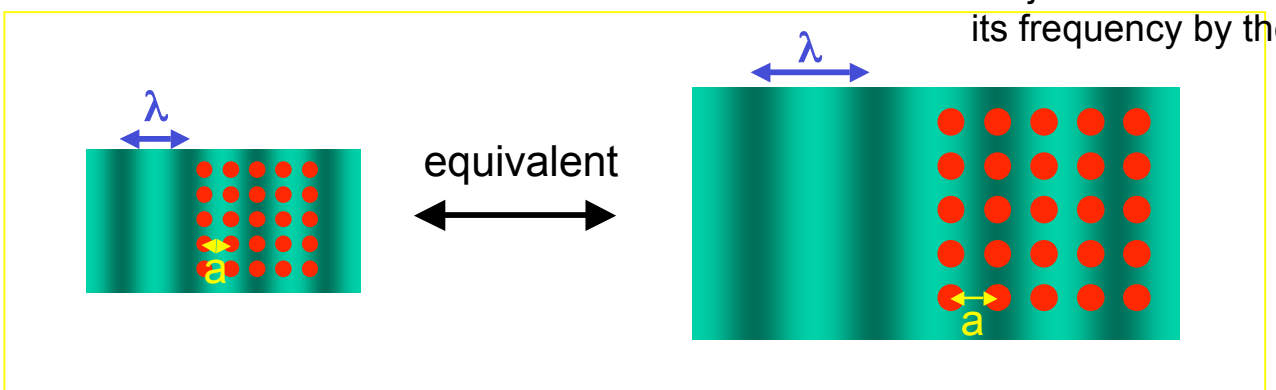
$$\nabla \times \frac{1}{\varepsilon(\mathbf{r})} \nabla \times \mathbf{H}(\mathbf{r}) = \left(\frac{\omega}{c}\right)^2 \mathbf{H}(\mathbf{r})$$

Then, in a configuration of dielectric ε'(r') that is just a compressed or expanded version of ε: ε'(r') = ε(r'/s), Using r' = sr, **H(r**'/s) = **H'(r**'), and

$$\nabla' = \nabla / s$$

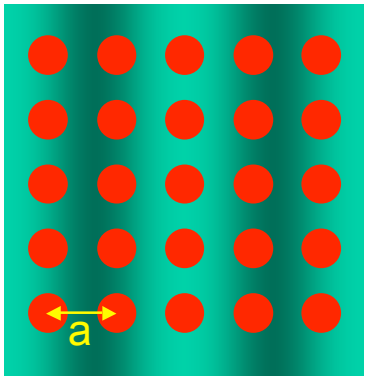
$$\nabla' \times \frac{1}{\varepsilon'(\mathrm{r}')} \nabla' \times \mathrm{H}'(\mathrm{r}') = \left(\frac{\omega}{cs}\right)^2 \mathrm{H}'(\mathrm{r}')$$

After changing the length scale by *s*,
we just scale the old mode and its frequency by the same factor



Normalized units

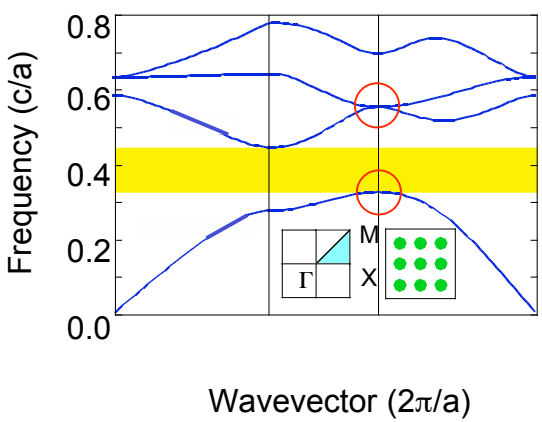
The lattice constant: *a*



The units of the following physical quantities become:

- Frequency: c/a
- Angular frequency: $2\pi c/a$
- Wavevector: $2\pi/a$
- Wavelength: a

A simple example for reading the band diagram



Gap extends from 0.2837 c/a to 0.4183 c/a
The mid gap frequency is at 0.3510 c/a

To design a crystal such that 1.55 micron light falls at the center of the gap, we have

c/(1.55micron) = 0.3510 c/a, hence
*a = 0.3510 * 1.55 micron = 0.5440 micron*

Electromagnetic energy and the variational principle

From the Master equations:

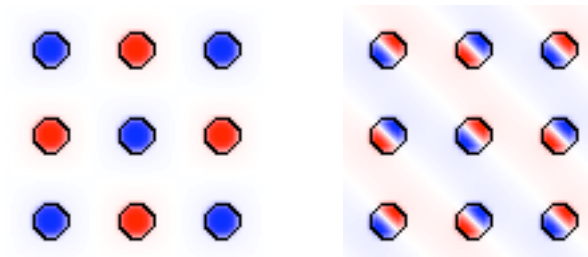
$$\Theta \mathrm{H}(\mathrm{r}) = \nabla \times \frac{1}{\varepsilon(\mathrm{r})} \nabla \times \mathrm{H}(\mathrm{r}) = \left(\frac{\omega}{c}\right)^2 \mathrm{H}(\mathrm{r})$$

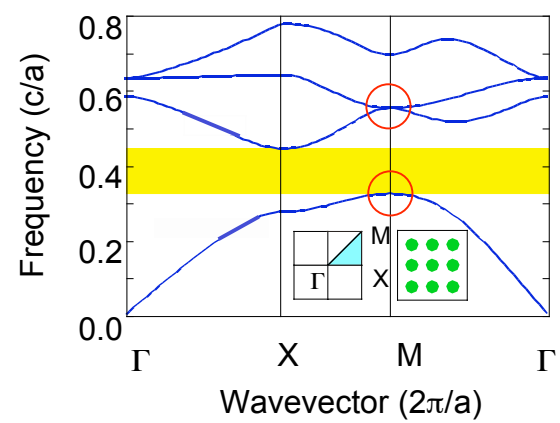
Integral form:

$$\left(\frac{\omega}{c}\right)^2 = \frac{\int d\mathrm{r} \frac{1}{\varepsilon(\mathrm{r})} \boxed{|\nabla \times \mathrm{H}(\mathrm{r})|^2}}{\int d\mathrm{r} |\mathrm{H}(\mathrm{r})|^2}$$

Concentration of the displacement field ***D*** in the high dielectric constant region minimizes the frequency.

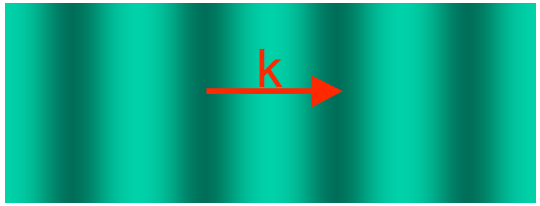
First two bands, at M-point
Displacement field





A simple example of the band-structure: vacuum (1d)

Vacuum: ε=1, μ=1, plane-wave solution to the Maxwell's equation:



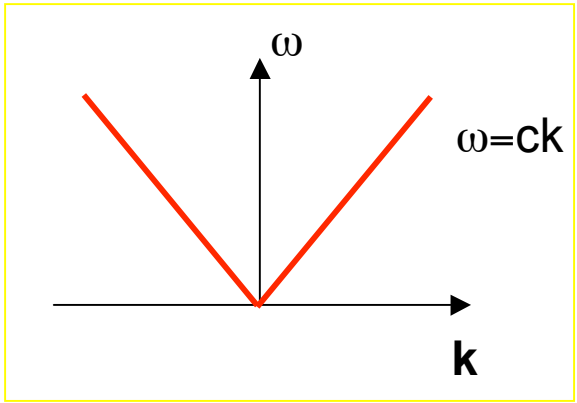
$$\mathbf{H}e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$$

with a transversality constraints: $\mathbf{k}\bullet\mathbf{H}=0$

A band structure, or dispersion relation defines the relation between the frequency ω, and the wavevector k.

$$\omega = c|\mathbf{k}|$$

For a one-dimensional system, the band structure can be simply depicted as:

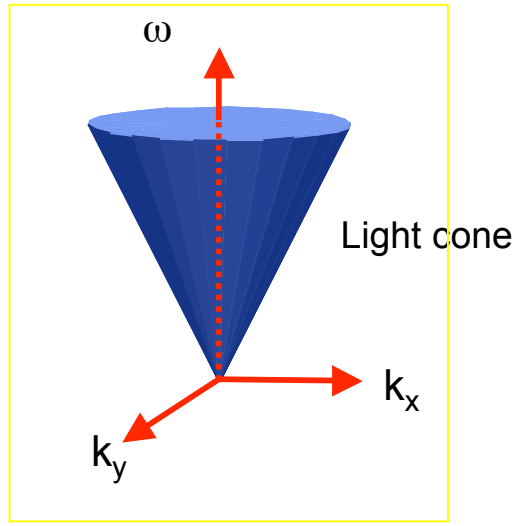


Visualization of the vacuum band structure (2d)

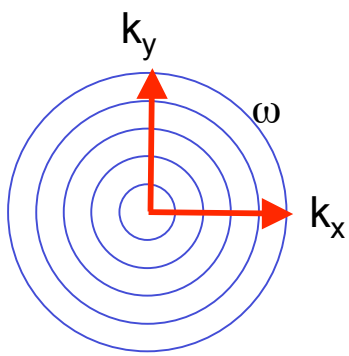
For a two-dimensional system:

$$\omega = c\sqrt{k_x^2 + k_y^2}$$

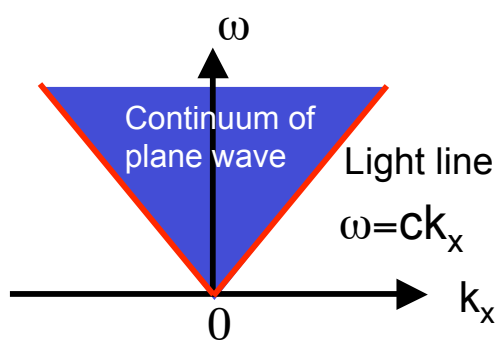
This function depicts a cone: light cone.



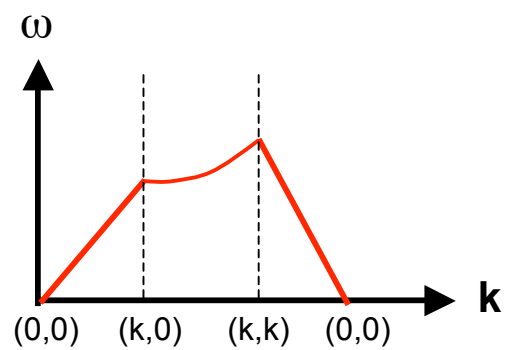
A few ways to visualize this band structure :



Constant frequency contour



Projected band diagram



Band diagram along several “special” directions

Electromagnetic energy and the variational principle

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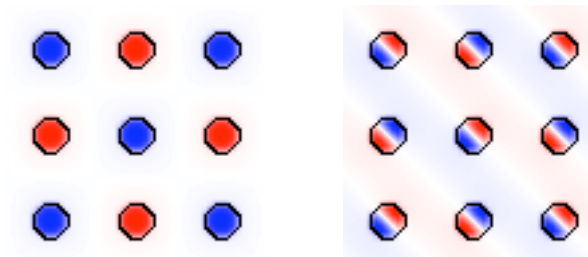

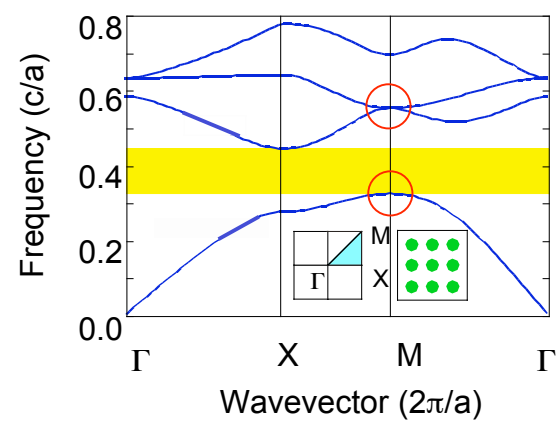

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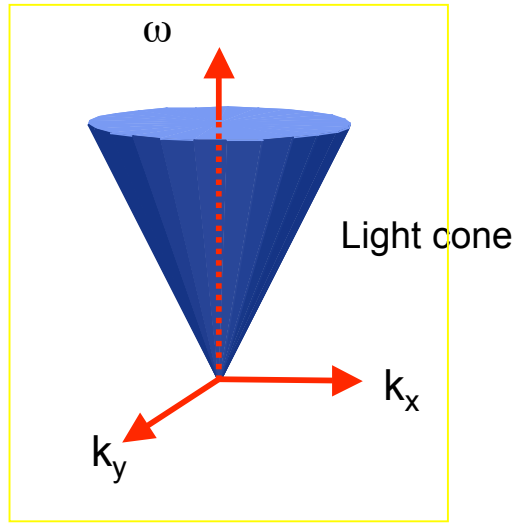

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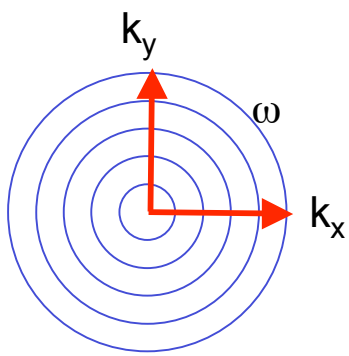

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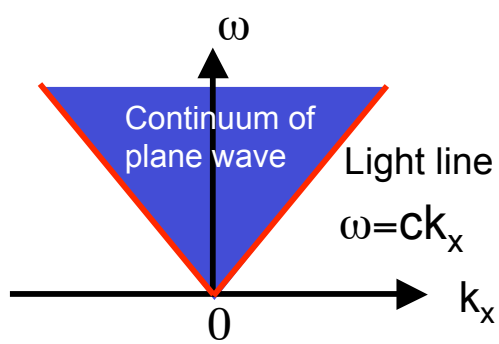
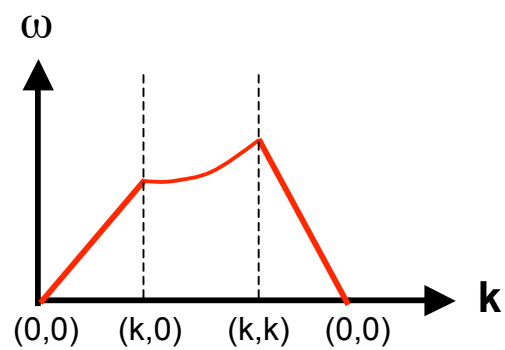


Projected band diagram

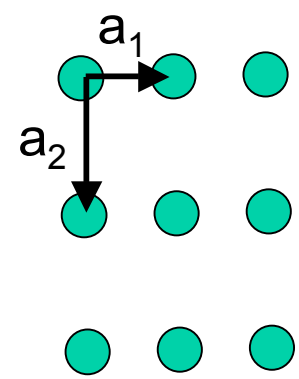


Band diagram along several “special” directions

Bloch theorem for electromagnetism

In a periodic dielectric media, i.e. $\varepsilon(r+a)=\varepsilon(r)$, the solution H(r) to the Master's equation:

$$\nabla \times \frac{1}{\varepsilon(\mathrm{r})} \nabla \times \mathrm{H}(\mathrm{r}) = \left(\frac{\omega}{c}\right)^2 \mathrm{H}(\mathrm{r})$$



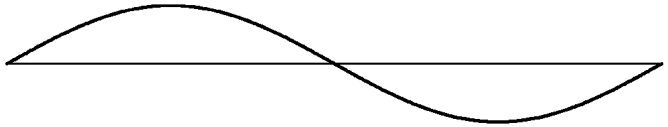
has to satisfy the following relations:

$$\mathrm{H}(\mathrm{r}) = e^{i(\mathrm{k}\cdot\mathrm{r})} \mathrm{u}_{\mathrm{k}}(\mathrm{r})$$

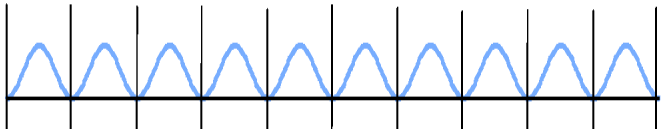
where $u_k(r) = u_k(r+a)$ is a periodic function.

Bloch wave functions

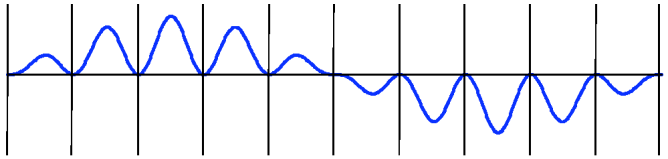
$$e^{ikx}$$



$$u(x)$$

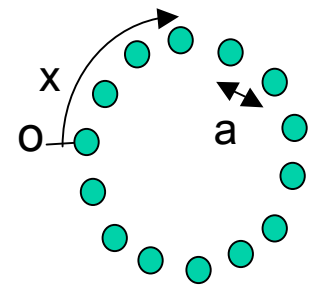


$$e^{ikx}u(x)$$



A simple proof of Bloch theorem *(Solid state Phys. Kittel, p179-180)*

Proof in 1 dimension



- Consider *N* identical lattice points on a ring of length *N*a
- The dielectric function is periodic in a, with $\varepsilon(x)=\varepsilon(x+sa)$, where s is an integer
- Translational symmetry ⇨ Expect solutions of the wave equation

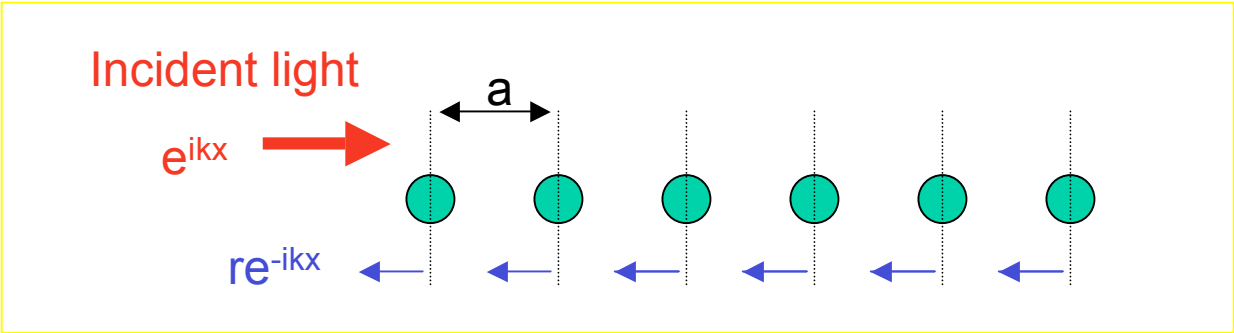
$$\mathbf{H}(x+a) = C\,\mathbf{H}(x)$$

- Going once around the ring: $\mathbf{H}(x+Na) = \mathbf{H}(x) \overset{\text{n-times}}{=} C^N\,\mathbf{H}(x)$

⇨ C is one of the N roots of unity: $C = \exp(i2\pi s/N)$; s = 0, 1, 2, …, N-1

- Bloch function $\begin{cases} \mathbf{H}(x) = \mathbf{u}_k(x)\exp(i2\pi sx/(Na)) \\ \text{Where } u_k(x+a) = u_k(x) \end{cases}$ satisfies $\begin{cases} \mathbf{H}(x+a) = C\,\mathbf{H}(x) \\ \mathbf{H}(x+Na) = \mathbf{H}(x) \end{cases}$

Bragg scattering



Regardless of how small the reflectivity r is from an individual scatter, the total reflection R from a semi infinite structure:

$$R = re^{-ikx} + re^{-2ika}e^{-ikx} + re^{-4ika}e^{-ikx} + \ldots = re^{-ikx}\frac{1}{1-e^{-2ika}}$$

Diverges if

$$\boxed{e^{2ika} = 1 \qquad k = \frac{\pi}{a}}$$

Bragg condition

Light can not propagate in a crystal, when the frequency of the incident light is such that the Bragg condition is satisfied → Origin of the photonic band gap

A brief review of the reciprocal lattice

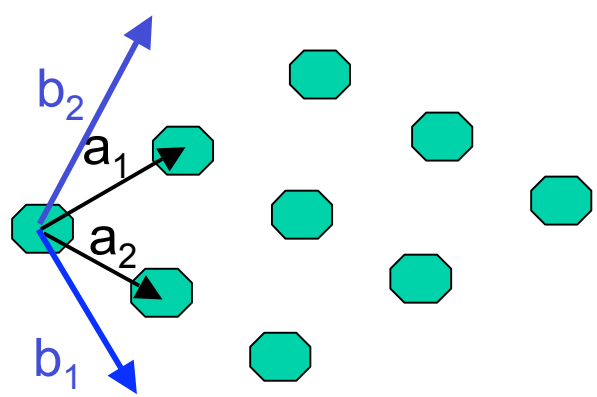
See Kittel, pp. 32-33

•The reciprocal lattice vector **G** is defined by: $\mathbf{e}^{i\mathbf{G}\cdot\mathbf{a}} = 0$
where a is *any* lattice vector of the crystal.

• For a given set of lattice vectors, $\mathbf{a}_1$, $\mathbf{a}_2$ and $\mathbf{a}_3$, the set of basis vectors for the reciprocal lattice is:

$$\mathbf{b}_1 = \frac{\mathbf{a}_2 \times \mathbf{a}_3}{\mathbf{a}_1 \bullet (\mathbf{a}_2 \times \mathbf{a}_3)} \cdot 2\pi \qquad \mathbf{b}_2 = \frac{\mathbf{a}_3 \times \mathbf{a}_1}{\mathbf{a}_1 \bullet (\mathbf{a}_2 \times \mathbf{a}_3)} \cdot 2\pi \qquad \mathbf{b}_3 = \frac{\mathbf{a}_1 \times \mathbf{a}_2}{\mathbf{a}_1 \bullet (\mathbf{a}_2 \times \mathbf{a}_3)} \cdot 2\pi$$

The reciprocal lattice vector **G** are: $\mathbf{G} = n_1\mathbf{b}_1 + n_2\mathbf{b}_2 + n_3\mathbf{b}_3$, where n_1, n_2, n_3 are arbitrary integers.



Summary

•Photonic crystals are artificial media with a periodic index contrast.

•Electromagnetic wave in a photonic crystal is described by a band structure, which relates the frequency of modes to the wavevectors.

•Fundamental properties of modes: scale invariance, orthogonality.