Lecture 1: A Brief Review of Maxwell’s Equations

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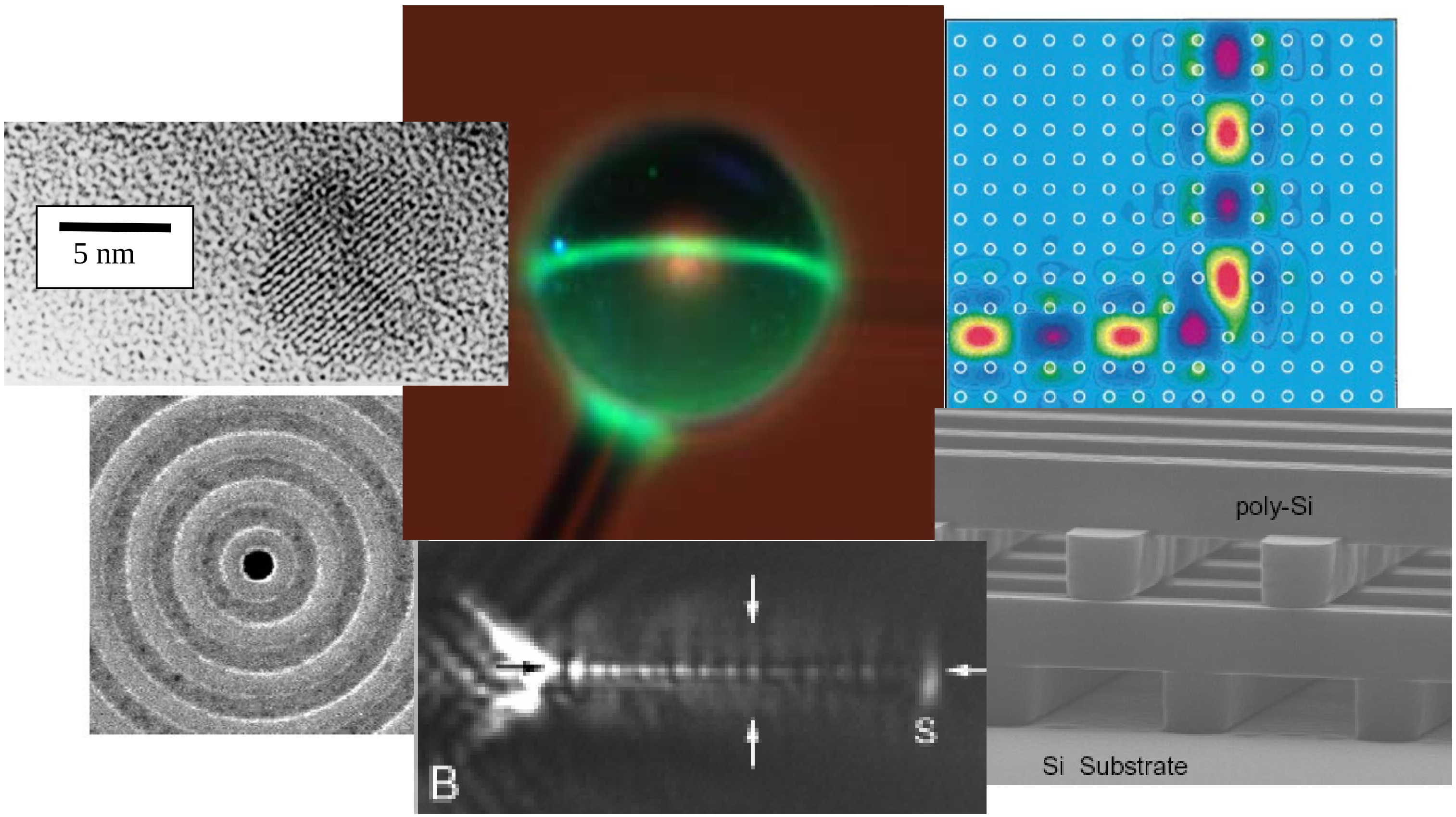
ECE695S



Light Interaction with Matter

Maxwell's Equations

$$\nabla \cdot \boldsymbol{D} = \rho_f$$

Divergence equations	Curl equations
	$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial \boldsymbol{t}}$
$\nabla \cdot \boldsymbol{B} = 0$	$\nabla \times \boldsymbol{H} = \frac{\partial \boldsymbol{D}}{\partial \boldsymbol{t}} + \boldsymbol{J}$

$\boldsymbol{D}$ = Electric flux density

$\boldsymbol{E}$ = Electric field vector

$\boldsymbol{\rho}$ = charge density

$\boldsymbol{B}$ = Magnetic flux density

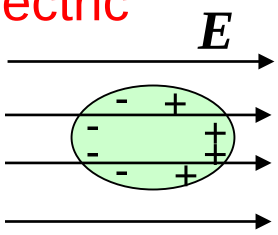
$\boldsymbol{H}$ = Magnetic field vector

$\boldsymbol{J}$ = current density

Constitutive Relations

Constitutive relations relate flux density to polarization of a medium

Electric



When $\boldsymbol{P}$ is proportional to $\boldsymbol{E}$

$$\boldsymbol{D} = \varepsilon_0 \boldsymbol{E} + \boldsymbol{P}\left(\boldsymbol{E}\right) = \varepsilon \boldsymbol{E}$$

Electric polarization vector…… Material dependent!!

ε_0 = Dielectric constant of vacuum = 8.85 · 10^{-12} $C^2N^{-1}m^{-2}$ [F/m]

ε = Material dependent dielectric constant

Total electric flux density = Flux from external E-field + flux due to material polarization

Magnetic

$$\boldsymbol{B} = \mu_0 \boldsymbol{H} + \mu_0 \boldsymbol{M}\left(\boldsymbol{H}\right)$$

Magnetic flux density   Magnetic field vector   Magnetic polarization vector

μ_0 = permeability of free space = $4\pi x10^{-7}$ H/m

Note: For now, we will focus on materials for which

$$\boldsymbol{M} = 0 \Rightarrow \boldsymbol{B} = \mu_0 \boldsymbol{H}$$

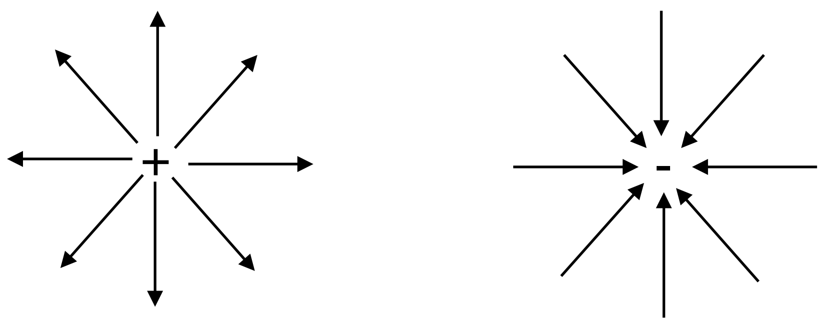
Divergence Equations

How did people come up with: $\nabla \cdot \boldsymbol{D} = \rho$?

Coulomb

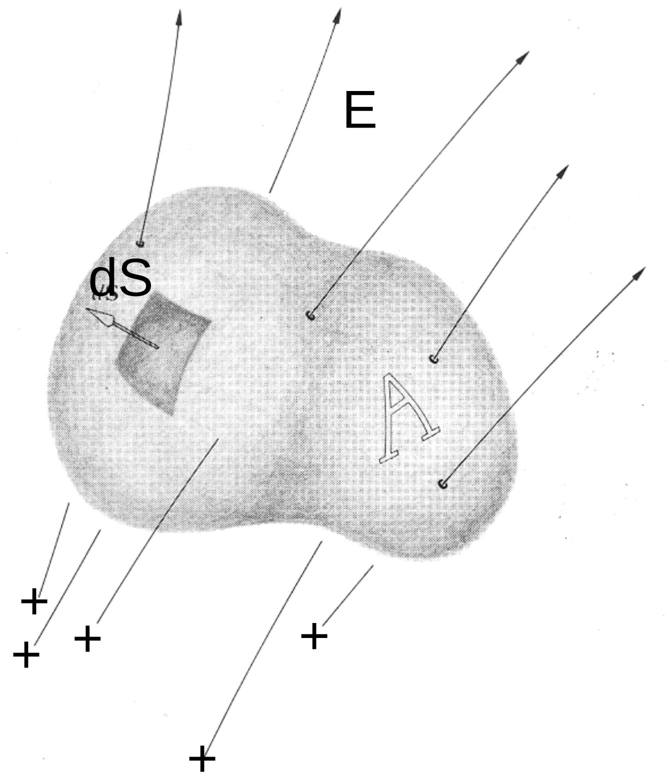
- Charges of same sign repel each other (+ and + or – and -)
- Charges of opposite sign attract each other (+ and -)
- He explained this using the concept of an electric field : $\vec{F} = q\vec{E}$

⇨ Every charge has some field lines associated with it



- He found: Larger charges give rise to stronger forces between charges
- Coulomb explained this with a stronger field (more field lines)

Divergence Equations



Gauss's Law (Gauss 1777-1855)

$$\int_A \boldsymbol{D} \cdot d\boldsymbol{S} = \int_A \varepsilon E \cdot d\boldsymbol{S} = \int_V \rho dv$$

E-field related to enclosed charge

Gauss's Theorem (very general)

$$\int_A \boldsymbol{F} \cdot d\boldsymbol{S} = \int_V \nabla \cdot \boldsymbol{F} dv$$

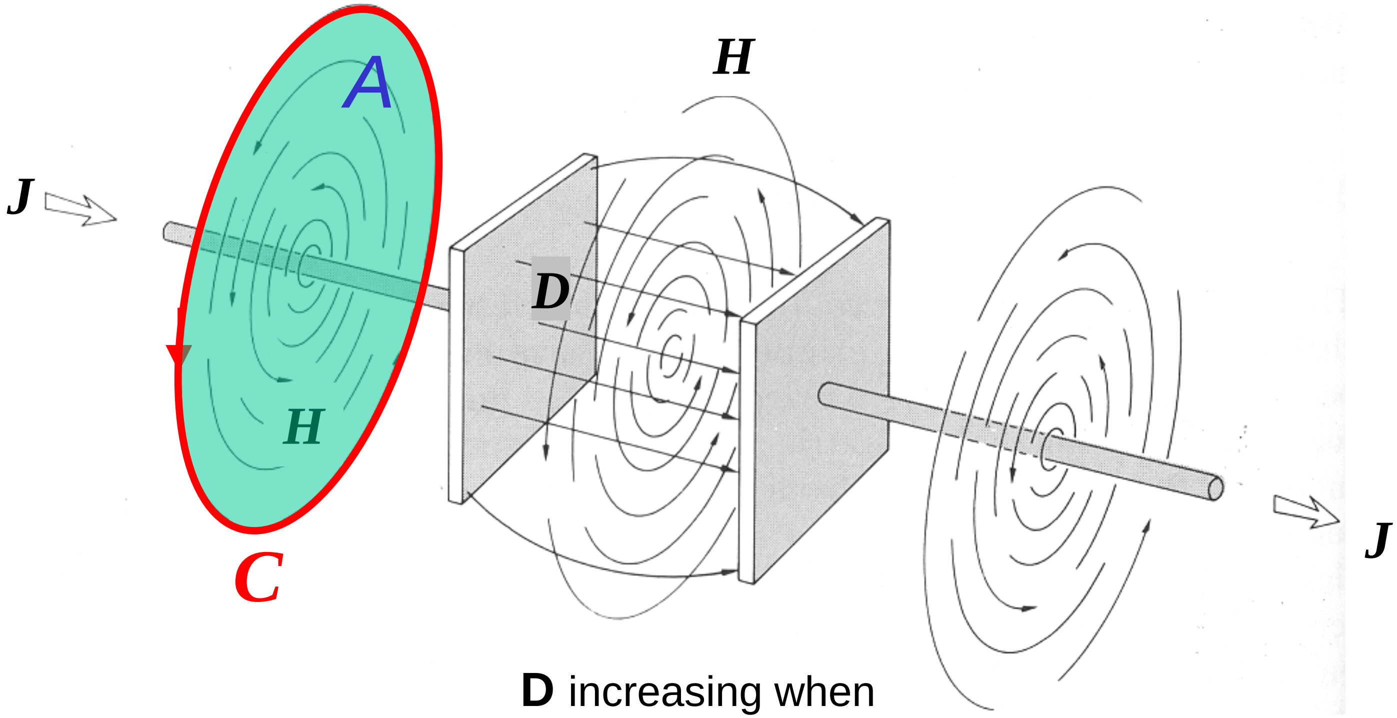
Combining the 2 Gauss's

$$\int_A \boldsymbol{D} \cdot d\boldsymbol{S} = \int_V \nabla \cdot \boldsymbol{D} dv = \int_V \rho dv \quad \Rightarrow \quad \nabla \cdot \boldsymbol{D} = \rho$$

The other divergence eq. $\nabla \cdot \boldsymbol{B} = 0$ is derived in a similar way from $\int_A \boldsymbol{B} \cdot d\boldsymbol{S} = 0$

Curl Equations

How did people come up with: $\nabla \times \boldsymbol{H} = \frac{\partial \boldsymbol{D}}{\partial t} + \boldsymbol{J}$?



D increasing when charging the capacitor

Ampere (1775-1836)

$$\oint_C \boldsymbol{H} \cdot d\boldsymbol{l} = \int_A \left(\frac{\partial \boldsymbol{D}}{\partial t} + \boldsymbol{J} \right) \cdot d\boldsymbol{S}$$

Magnetic field induced by:
- Changes in el. flux
- Electrical currents

Curl Equations

Ampere: $\oint_C \boldsymbol{H}\cdot d\boldsymbol{l} = \int_A \left(\frac{\partial \boldsymbol{D}}{\partial t} + \boldsymbol{J}\right)\cdot d\boldsymbol{S}$

Stokes theorem: $\oint_C \boldsymbol{F}\cdot d\boldsymbol{l} = \int_A \left(\nabla\times\boldsymbol{F}\right)\cdot d\boldsymbol{S}$

$$\oint_C \boldsymbol{H}\cdot d\boldsymbol{l} = \int_A \underline{\left(\nabla\times\boldsymbol{H}\right)}\cdot d\boldsymbol{S} = \int_A \underline{\left(\frac{\partial \boldsymbol{D}}{\partial t} + \boldsymbol{J}\right)}\cdot d\boldsymbol{S}$$

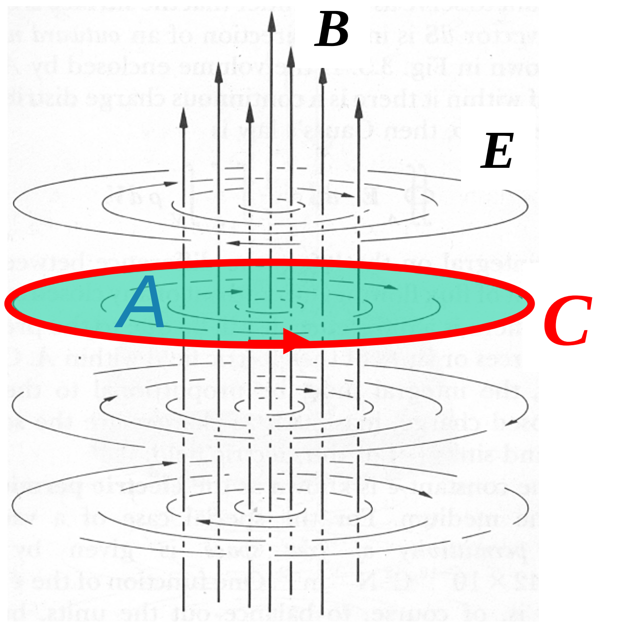
$$\nabla\times\boldsymbol{H} = \frac{\partial \boldsymbol{D}}{\partial t} + \boldsymbol{J}$$

$\nabla\times\boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$ Other curl eq.

Derived in a similar way from $\oint_C \boldsymbol{E}\cdot d\boldsymbol{l} = -\int_A \frac{\partial \boldsymbol{B}}{\partial t}\cdot d\boldsymbol{S}$

$$\oint_C \boldsymbol{E}\cdot d\boldsymbol{l} = \int_A \underline{\left(\nabla\times\boldsymbol{E}\right)}\cdot d\boldsymbol{S} = -\int_A \underline{\frac{\partial \boldsymbol{B}}{\partial t}}\cdot d\boldsymbol{S}$$

Stokes



Summary Maxwell's Equations

Divergence equations

$$\nabla \cdot \boldsymbol{D} = \rho$$

$$\nabla \cdot \boldsymbol{B} = 0$$

Flux lines start and end on charges or poles

Curl equations

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial \boldsymbol{t}}$$

$$\nabla \times \boldsymbol{H} = \frac{\partial \boldsymbol{D}}{\partial \boldsymbol{t}} + \boldsymbol{J}$$

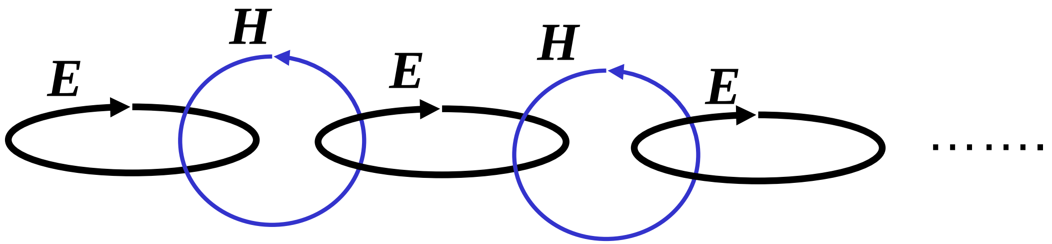
Changes in fluxes give rise to fields

Currents give rise to H-fields

Note: No constants such as μ_0 $\varepsilon_{0,}$ μ ε, c, χ,……. appear when Eqs are written this way.

The Wave Equation

Plausibility argument for existence of EM waves



Curl equations: Changing $\boldsymbol{E}$-field results in changing $\boldsymbol{H}$-field results in changing $\boldsymbol{E}$- field….

The real thing

Goal: Derive a wave equation: $\nabla^2 \boldsymbol{U}(r,t) = \frac{1}{v^2}\frac{\partial^2 \boldsymbol{U}(r,t)}{\partial \boldsymbol{t}^2}$ for $\boldsymbol{E}$ and $\boldsymbol{H}$

Solution: Waves propagating with a (phase) velocity v

$$\boldsymbol{U}(\boldsymbol{r},t) = \mathrm{Re}\left\{\boldsymbol{U}_0(\boldsymbol{r})\exp(i\omega t)\right\}$$

Position Time

Starting point: The curl equations

The Wave Equation for the E-field

Goal: $$\nabla^2 \boldsymbol{E}(r,t) = \frac{1}{v^2}\frac{\partial^2 \boldsymbol{E}(r,t)}{\partial \boldsymbol{t}^2}$$

Curl Eqs: a) $\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial \boldsymbol{t}} = -\mu_0 \frac{\partial \boldsymbol{H}}{\partial t}$ (Materials with $\boldsymbol{M}$ = 0 only)

b) $\nabla \times \boldsymbol{H} = \frac{\partial \boldsymbol{D}}{\partial \boldsymbol{t}} + \boldsymbol{J}$

Step 1: Try and obtain partial differential equation that just depends on $\boldsymbol{E}$

Apply curl on both side of a)

$$\nabla \times \nabla \times \boldsymbol{E} = \nabla \times \left(-\mu_0 \frac{\partial \boldsymbol{H}}{\partial t} \right) = -\mu_0 \frac{\partial \left(\nabla \times \boldsymbol{H} \right)}{\partial t}$$

Step 2: Substitute b) into a)

$$\nabla \times \nabla \times \boldsymbol{E} = -\mu_0 \frac{\partial^2 \boldsymbol{D}}{\partial t^2} - \mu_0 \frac{\partial \boldsymbol{J}}{\partial t} = -\mu_0 \varepsilon_0 \frac{\partial^2 \boldsymbol{E}}{\partial t^2} - \mu_0 \frac{\partial^2 \boldsymbol{P}}{\partial t^2} - \mu_0 \frac{\partial \boldsymbol{J}}{\partial t}$$

$\boldsymbol{D} = \varepsilon_0 \boldsymbol{E} + \boldsymbol{P}$

Cool!....looks like a wave equation already

The Wave Equation for the E-field

Compare: $$\nabla^2 \boldsymbol{E} = \frac{1}{v^2}\frac{\partial^2 \boldsymbol{E}}{\partial \boldsymbol{t}^2}$$

With: $$\nabla \times \nabla \times \boldsymbol{E} = -\mu_0 \varepsilon_0 \frac{\partial^2 \boldsymbol{E}}{\partial t^2} - \mu_0 \frac{\partial^2 \boldsymbol{P}}{\partial t^2} - \mu_0 \frac{\partial \boldsymbol{J}}{\partial t}$$

Use vector identity: $$\nabla \times \nabla \times \boldsymbol{E} = \nabla\left(\nabla \cdot \boldsymbol{E}\right) - \nabla^2 \boldsymbol{E}$$!

Verify that $\nabla \cdot \boldsymbol{E} = 0$ when *1)* $\rho_f = 0$

2) ε(r) does not vary significantly within a λ distance

Result: $$\nabla^2 \boldsymbol{E} = \mu_0 \varepsilon_0 \frac{\partial^2 \boldsymbol{E}}{\partial t^2} + \mu_0 \frac{\partial^2 \boldsymbol{P}}{\partial t^2} + \mu_0 \frac{\partial \boldsymbol{J}}{\partial t}$$

In order to solve this we need: 1) Find ***P(E)***

2) Find ***J(E)…***something like Ohm's law: ***J(E)*** = ***σE***

… we will look at this later..for now assume: ***J(E)*** = 0

Dielectric Media

Linear, Homogeneous, and Isotropic Media

$\boldsymbol{P}$ linearly proportional to $\boldsymbol{E}$: $\boldsymbol{P} = \varepsilon_0 \chi \boldsymbol{E}$

χ is a scalar constant called the "*electric susceptibility*"

$$\nabla^2 \boldsymbol{E} = \mu_0 \varepsilon_0 \frac{\partial^2 \boldsymbol{E}}{\partial t^2} + \mu_0 \frac{\partial^2 \boldsymbol{P}}{\partial t^2} + \cancel{\mu_0 \frac{\partial \boldsymbol{J}}{\partial t}}$$

$$\nabla^2 \boldsymbol{E} = \mu_0 \varepsilon_0 \frac{\partial^2 \boldsymbol{E}}{\partial t^2} + \mu_0 \varepsilon_0 \chi \frac{\partial^2 \boldsymbol{E}}{\partial t^2} = \mu_0 \varepsilon_0 (1 + \chi) \frac{\partial^2 \boldsymbol{E}}{\partial t^2}$$

Define relative dielectric constant as: $\varepsilon_r = 1 + \chi$

All the materials properties

$$\nabla^2 \boldsymbol{E} = \mu_0 \varepsilon_0 \varepsilon_r \frac{\partial^2 \boldsymbol{E}}{\partial t^2}$$

Results from $\boldsymbol{P}$

Note 1 : In anisotropic media $\boldsymbol{P}$ and $\boldsymbol{E}$ are not necessarily parallel: $P_i = \sum_j \varepsilon_0 \chi_{ij} E_j$

Note2 : In non-linear media: $\boldsymbol{P} = \varepsilon_0 \chi \boldsymbol{E} + \varepsilon_0 \chi^{(2)} \boldsymbol{E}^2 + \varepsilon_0 \chi^{(3)} \boldsymbol{E}^3 + \ldots\ldots$

Properties of EM Waves in Bulk Materials

We have derived a wave equation for EM waves!

$$\boxed{\nabla^2 \boldsymbol{E} = \mu_0 \varepsilon_0 \varepsilon_r \frac{\partial^2 \boldsymbol{E}}{\partial t^2}}$$

Electromagnetic Waves

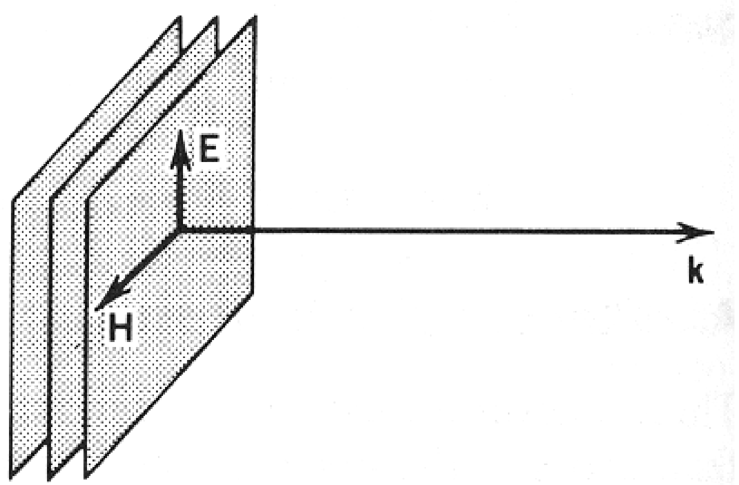
Solution to: $$\nabla^2 \boldsymbol{E}(\boldsymbol{r},t) = \frac{n^2}{c^2}\frac{\partial^2 \boldsymbol{E}(\boldsymbol{r},t)}{\partial \boldsymbol{t}^2}$$

Monochromatic waves: $$\boldsymbol{E}(\boldsymbol{r},t) = \mathrm{Re}\left\{\boldsymbol{E}(\boldsymbol{k},\omega)\exp\left(-i\boldsymbol{k}\cdot\boldsymbol{r} + i\omega t\right)\right\}$$

$$\boldsymbol{H}(\boldsymbol{r},t) = \mathrm{Re}\left\{\boldsymbol{H}(\boldsymbol{k},\omega)\exp\left(-i\boldsymbol{k}\cdot\boldsymbol{r} + i\omega t\right)\right\}$$

Check these are solutions!

TEM wave



Symmetry Maxwell's Equations result in $\boldsymbol{E} \perp \boldsymbol{H} \perp$ propagation direction

Optical intensity

Time average of Poynting vector: $\boldsymbol{S}(\boldsymbol{r},t) = \boldsymbol{E}(\boldsymbol{r},t)\times\boldsymbol{H}(\boldsymbol{r},t)$

Speed of an EM Wave in Matter

Speed of the EM wave:

Compare $\nabla^2 \boldsymbol{E} = \mu_0 \varepsilon_0 \varepsilon_r \frac{\partial^2 \boldsymbol{E}}{\partial t^2}$ and $\nabla^2 \boldsymbol{E} = \frac{1}{v^2} \frac{\partial^2 \boldsymbol{E}}{\partial \boldsymbol{t}^2}$

$$\Rightarrow \quad v^2 = \frac{1}{\mu_0 \varepsilon_0} \frac{1}{\varepsilon_r} = \frac{c_0^2}{\varepsilon_r}$$

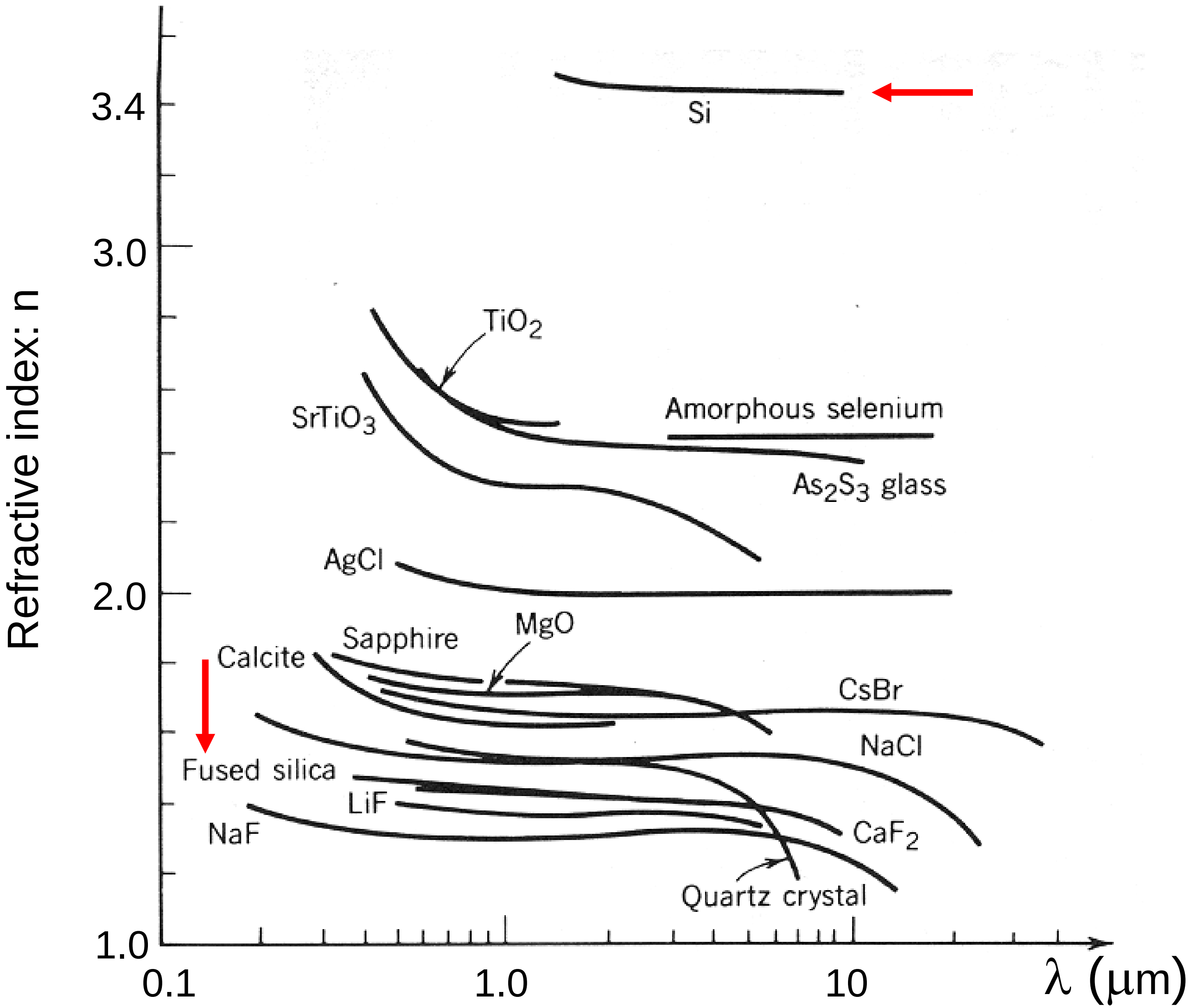
Where $c_0^2 = 1/(\varepsilon_0\ \mu_0) = 1/((8.85\text{x}10^{-12}\ \text{C}^2/\text{m}^3\text{kg})\ (4\pi\ \text{x}\ 10^{-7}\ \text{m kg/C}^2)) = (3.0\ \text{x}\ 10^8\ \text{m/s})^2$

Optical refractive index

Refractive index is defined by: $n = \frac{c}{v} = \sqrt{\varepsilon_r} = \sqrt{1+\chi}$

Note: Including polarization results in same wave equation with a different ε_r $\Rightarrow$ c becomes v

Refractive Index Various Materials



Dispersion Relation

Dispersion relation: ω = ω(k)

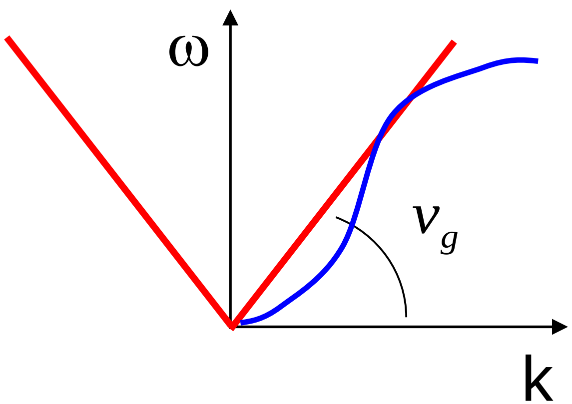
Derived from wave equation $\nabla^2 \boldsymbol{E}(\boldsymbol{r},t) = \frac{n^2}{c^2}\frac{\partial^2 \boldsymbol{E}(\boldsymbol{r},t)}{\partial \boldsymbol{t}^2}$

Substitute: $\boldsymbol{E}(z,t) = \mathrm{Re}\left\{\boldsymbol{E}(z,\omega)\exp(-ikr + i\omega t)\right\}$

Result: $k^2 = \frac{n^2}{c^2}\omega^2$

Check this!

$$\omega^2 = \frac{c^2}{n^2}k^2$$



Group velocity: $v_g \equiv \frac{d\omega}{dk}$

Phase velocity: $v_{ph} = \frac{\omega}{k} = \frac{c}{n} = \frac{c}{\sqrt{\varepsilon_r}} = \frac{c}{\sqrt{1+\chi}}$

Absorption and Dispersion of EM Waves

Transparent materials can be described by a purely real refractive index n

EM wave: $\boldsymbol{E}(z,t)=\mathrm{Re}\left\{\boldsymbol{E}(z,\omega)\exp\left(-ikz+i\omega t\right)\right\}$

Dispersion relation $\omega^2=\dfrac{c^2}{n^2}k^2 \Rightarrow k=\pm\dfrac{\omega}{c}n$

Absorbing materials can be described by a complex n: $\boxed{n=n'+in''}$

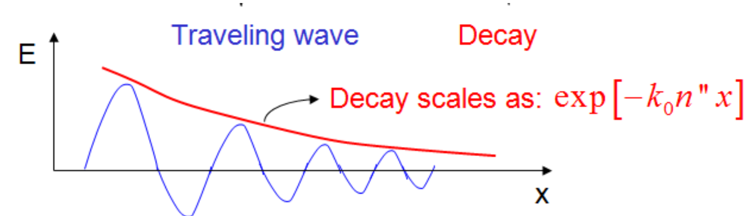
It follows that: $k=\pm\dfrac{\omega}{c}\left(n'+in''\right)=\pm\left(\dfrac{\omega}{c}n'+i\dfrac{\omega}{c}n''\right)\equiv\pm\left(\beta-i\dfrac{\alpha}{2}\right)$

Investigate + sign: $\boldsymbol{E}(z,t)=\mathrm{Re}\left\{\boldsymbol{E}(z,\omega)\exp\left(-i\beta z-\dfrac{\alpha}{2}z+i\omega t\right)\right\}$

Traveling wave Decay

Note: $\beta=\dfrac{\omega}{c}n'=kn'$ $\Rightarrow$ n' act as a regular refractive index

$\alpha=-2\dfrac{\omega}{c}n''=-2kn''$ $\Rightarrow$ α is the absorption coefficient



Summary

Maxwell's Equations

$$\nabla \cdot \boldsymbol{D} = \rho_f \qquad \nabla \cdot \boldsymbol{B} = 0 \qquad \nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial \boldsymbol{t}} \qquad \nabla \times \boldsymbol{H} = \frac{\partial \boldsymbol{D}}{\partial \boldsymbol{t}} + \boldsymbol{J}$$

Curl Equations lead to

$$\nabla^2 \boldsymbol{E} = \mu_0 \varepsilon_0 \frac{\partial^2 \boldsymbol{E}}{\partial t^2} + \mu_0 \frac{\partial^2 \boldsymbol{P}}{\partial t^2}$$ (under certain conditions)

Linear, Homogeneous, and Isotropic Media

$$\boldsymbol{P} = \varepsilon_0 \chi \boldsymbol{E}$$

Wave Equation with v = c/n

$$\nabla^2 \boldsymbol{E}(\boldsymbol{r}, t) = \frac{n^2}{c^2} \frac{\partial^2 \boldsymbol{E}(\boldsymbol{r}, t)}{\partial \boldsymbol{t}^2}$$

In real life: Relation between ***P*** and ***E*** is dynamic

$$\boldsymbol{P}(\boldsymbol{r}, t) = \varepsilon_0 \int_{-\infty}^{+\infty} dt' \, x(t - t') \boldsymbol{E}(\boldsymbol{r}, t') \quad \Rightarrow \quad \boldsymbol{P}(\boldsymbol{k}, \omega) = \varepsilon_0 \chi(\omega) \boldsymbol{E}(\boldsymbol{k}, \omega)$$

This will have major consequences !!!

Next 2 Lectures

Real and imaginary part of χ are linked

- Kramers-Kronig

- Origin frequency dependence of χ in real materials

Derivation of χ for a range of materials

- Insulators (Lattice absorption, Urbach tail, color centers…)

- Semiconductors (Energy bands, excitons …)

- Metals (Plasmons, plasmon-polaritons, …)

Useful Equations and Valuable Relations

Maxwell's Equations

Divergence Equations $\nabla \cdot \boldsymbol{D} = \rho$

$\nabla \cdot \boldsymbol{B} = 0$

Curl Equations $\nabla \times \boldsymbol{H} = \dfrac{\partial \boldsymbol{D}}{\partial \boldsymbol{t}} + \boldsymbol{J}$

$\nabla \times \boldsymbol{E} = -\dfrac{\partial \boldsymbol{B}}{\partial \boldsymbol{t}}$

Constitutive relations:

$$\boldsymbol{D} = \varepsilon_0 \boldsymbol{E} + \boldsymbol{P} = \varepsilon_0 + \varepsilon_0 \chi \boldsymbol{E} = \varepsilon_0 (1+\chi) \boldsymbol{E} = \varepsilon_o \varepsilon_r \boldsymbol{E}$$

$$\boldsymbol{B} = \mu_0 \boldsymbol{H} + \mu_0 \boldsymbol{M} = \mu_0 \boldsymbol{H} + \mu_0 \chi_m \boldsymbol{H} = \mu_0 (1+\chi_m) \boldsymbol{H} = \mu_0 \mu_r \boldsymbol{H}$$

Gauss's Law

$$\int_A \boldsymbol{D} \cdot d\boldsymbol{S} = \int_A \varepsilon E \cdot d\boldsymbol{S} = \int_V \rho dv$$

Maxwell (also)

$$\oint_C \boldsymbol{H} \cdot d\boldsymbol{l} = \int_A \left(\frac{\partial \boldsymbol{D}}{\partial t} + \boldsymbol{J} \right) \cdot d\boldsymbol{S}$$

Dynamic relation between P and E:

$$\boldsymbol{P}(\boldsymbol{r},t) = \varepsilon_0 \int_{-\infty}^{+\infty} dt' x(t-t') \boldsymbol{E}(\boldsymbol{r},t') \quad \text{and} \quad \boldsymbol{P}(\boldsymbol{k},\omega) = \varepsilon_0 \chi(\omega) \boldsymbol{E}(\boldsymbol{k},\omega)$$

Dispersive and absorbing materials:

$$\boldsymbol{E}(z,t) = \mathrm{Re}\left\{ \boldsymbol{E}(z,\omega) \exp\left(-i\beta z - \frac{\alpha}{2} z + i\omega t \right) \right\}$$

where $\beta = \dfrac{\omega}{c} n' = k_0 n'$,absorption coefficient $\alpha = -2\dfrac{\omega}{c} n'' = -2k_0 n''$

Handy Math Rules

Vector identities: $\nabla \times \nabla \times \boldsymbol{E} = \nabla(\nabla \cdot \boldsymbol{E}) - \nabla^2 \boldsymbol{E}$

$\nabla \cdot \varepsilon \boldsymbol{E} = \varepsilon \nabla \cdot \boldsymbol{E} + \boldsymbol{E} \cdot \nabla \varepsilon$

Gauss theorem $\displaystyle\int_A \boldsymbol{F} \cdot d\boldsymbol{S} = \int_V \nabla \cdot \boldsymbol{F} dv$

Stokes $\displaystyle\oint_C \boldsymbol{F} \cdot d\boldsymbol{l} = \int_A (\nabla \times \boldsymbol{F}) \cdot d\boldsymbol{S}$

Absorption and Dispersion of EM Waves

n is derived quantity from χ (next lecture we determine χ for different materials)

Complex n results from a complex χ:

$$\left.\begin{aligned} \chi &= \chi' + i\chi'' \\ n &= \sqrt{1+\chi} \end{aligned}\right\} \Rightarrow$$

$$\Rightarrow \left.\begin{aligned} n = n' + in'' = \sqrt{1+\chi} &= \sqrt{1+\chi'+i\chi''} \\ \alpha &= -2k_0 n'' \end{aligned}\right\} \Rightarrow n = n' - i\frac{\alpha}{2k_0} = \sqrt{1+\chi'+i\chi''}$$

Weakly absorbing media

When $\chi' \ll 1$ and $\chi'' \ll 1$:

$$\sqrt{1+\chi'+i\chi''} \approx 1 + \frac{1}{2}\left(\chi' + i\chi''\right)$$

Refractive index:

$$n' = 1 + \frac{1}{2}\chi'$$

Absorption coefficient: $\alpha = -2k_0 n'' = -k_0 \chi''$