ME 517: Micro- and Nanoscale Processes

Lecture 12: Atomic Force Microscopy - Cantilever II

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Cantilever Deflection Fundamentals

- Bernoulli-Euler Beam Flexure Formula

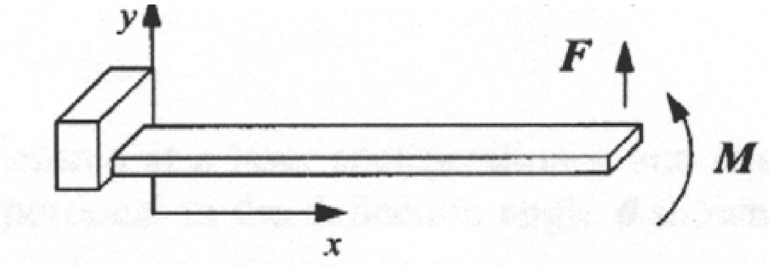
δ =deflection distance, x=position, M=applied moment, F=applied force, E=Young's modulus, I=moment of inertia

$$\frac{d^2\delta}{dx^2} = \frac{M(x)}{EI} \quad \text{Or equivalently} \quad \frac{d^4\delta}{dx^4} = \frac{F(x)}{EI}$$

- Subject to boundary conditions:

–Fixed left side (x=0): δ=0, $d\delta/dx$=0

–Free right side (x=L): δ =?, $d^2\delta/dx^2=0$, $d^3\delta/dx^3=0$



Cantilever Fundamentals Cont'd.

- For beams with constant cross sections and compositions, these equations integrate to:

$$EI\delta\left(x\right)=\int_0^x dx\int_0^x \boldsymbol{M}\left(x\right)dx+C_1x+C_2$$

$$EI\delta\left(x\right)=\int_0^x dx\int_0^x dx\int_0^x dx\int_0^x F\left(x\right)dx+C_1x^3+C_2x^2+C_3x+C_4$$

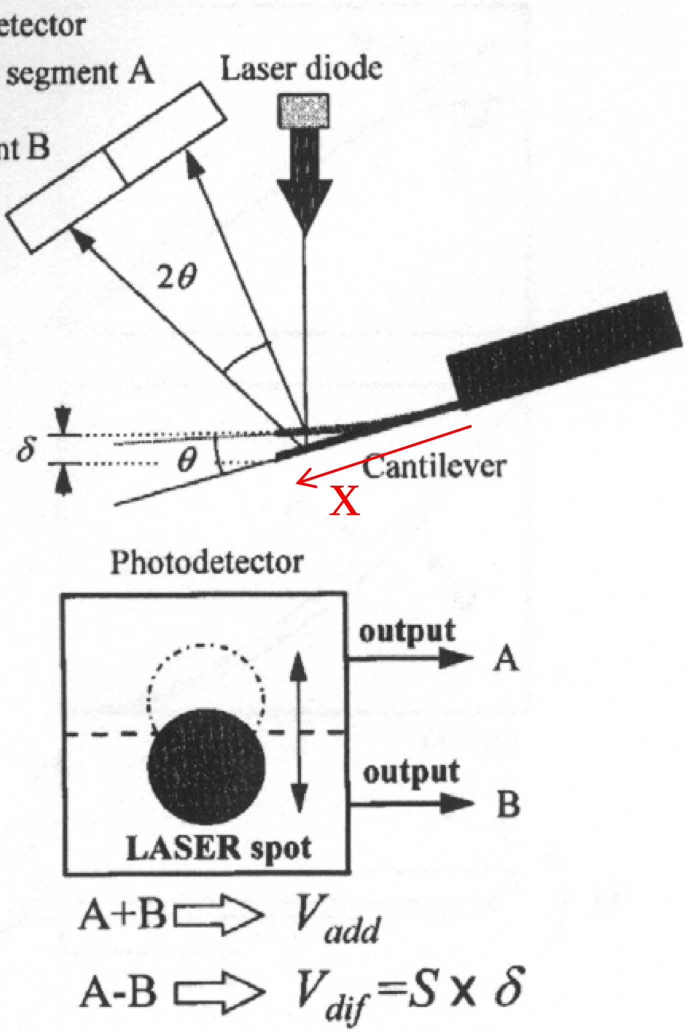
Rectangular Static Cantilever with Force at Tip

- L=length, W=width, t=thickness, I=moment of inertia, E=Young's modulus, ν=Poisson's ratio, F=force at tip, a=x/L

$$\theta_F = \frac{Fx}{2EI}\left(2L - x\right) = \frac{6FL^2}{EWt^3}a\left(2 - a\right)$$

$$\delta_F = \frac{Fx^2}{6EI}\left(3L - x\right) = \frac{2FL^3}{EWt^3}a^2\left(3 - a\right)$$

Cantilever Readout Config



Can show geometrically that for small angles,

$$\frac{d\delta}{dx} = \theta$$

Rectangular Static Cantilever with Force at Tip

- Calculate maximum displacement and angles at tip

$$\theta_{F\max} = \frac{6FL^2}{EWt^3}, \delta_{F\max} = \frac{4FL^3}{EWt^3}$$

- Substitute into previous relations

$$\theta_F = \frac{3\delta_{F\max}}{2L} a(2-a)$$

$$\delta_F = \frac{\delta_{F\max}}{2} a^2(3-a)$$

Rectangular Static Cantilever with Surface Coating

- Cantilever can be coated with thin film that has different properties from cantilever
- Results in surface stress difference $\Delta\sigma$ between top and bottom surfaces
- $\Delta\sigma$ is equivalent to moment applied at end of cantilever as

$$M = \frac{\Delta\sigma W t}{2}$$

Rectangular Static Cantilever with Surface Coating

- As before we can write angle and deflection

$$\theta_M = \frac{Mx}{EI} = \frac{12ML}{EWt^3}a \qquad \delta_M = \frac{Mx^2}{2EI} = \frac{6ML^2}{EWt^3}a^2$$

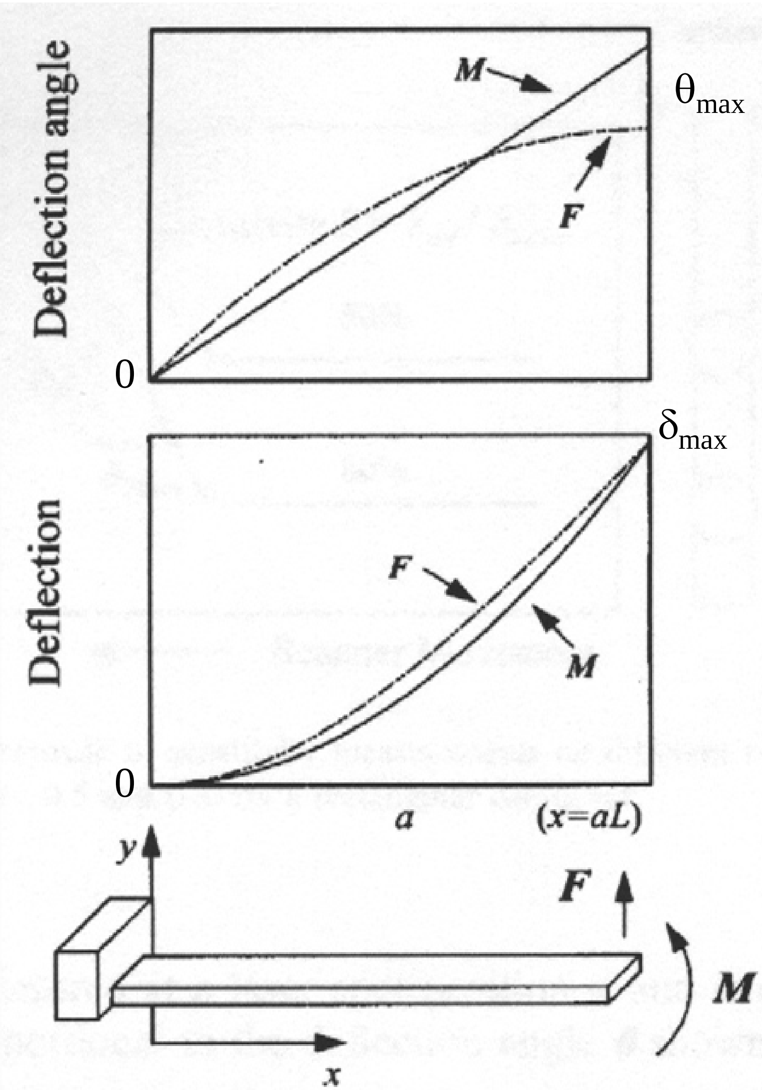
- With maximal values

$$\theta_{M\,\mathrm{max}} = \frac{12ML}{EWt^3} \qquad \delta_{M\,\mathrm{max}} = \frac{6ML^2}{EWt^3}$$

- Which becomes

$$\theta_M = \frac{2\delta_{M\,\mathrm{max}}}{L}a \qquad \delta_M = \delta_{M\,\mathrm{max}}a^2$$

Comparison of Deflections



Conclusions

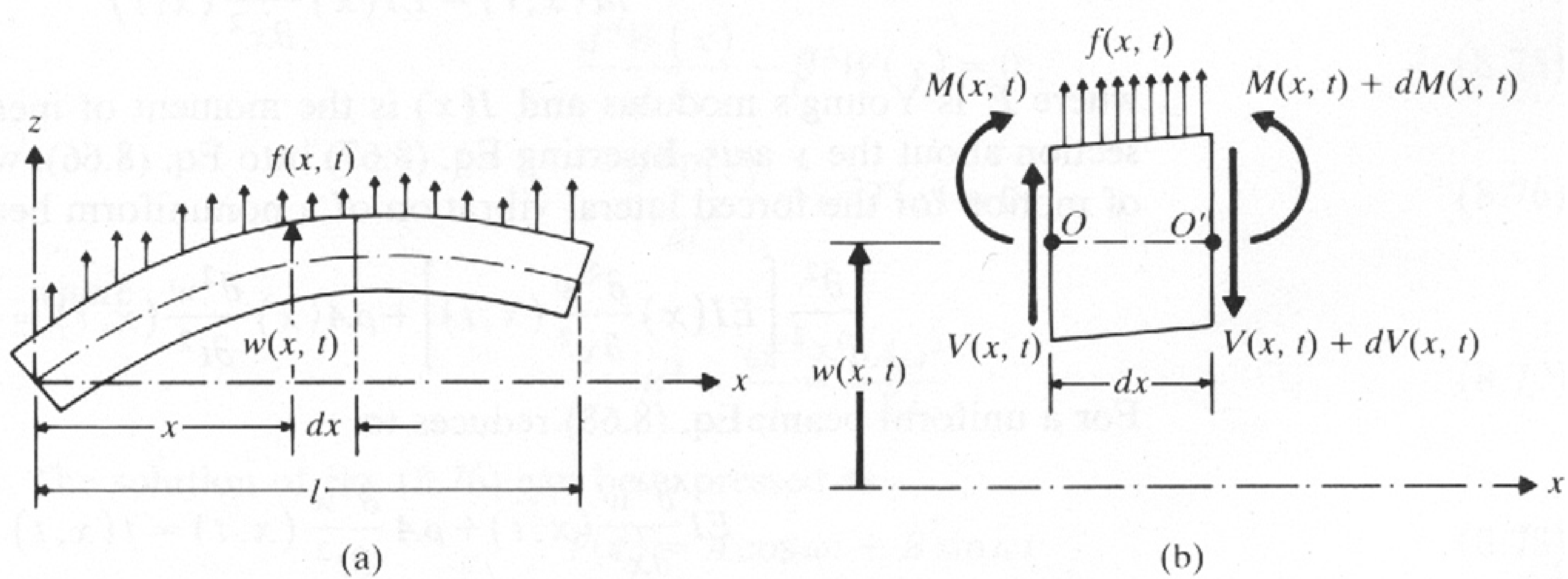
•Can have both imposed force and moment simultaneously—e.g. residual stresses

•Deflection angle is proportional to distance from base for surface stress loading

Dynamics of Cantilevers

- When used in tapping mode AFM, cantilever is vibrated at resonance
- Need to calculate dynamics of cantilevers

Variable Definition



•M(x,t) is bending moment, f(x,t) is external force per unit length, V(x,t) is shear force, w(x,t) is local height of cantilever above some reference

Mechanical Vibrations, Rao (*Purdue*), 1986.

Summing Forces and Moments

the force equation of motion in the z direction gives

ΣF=ma: $$-(V+dV)+f(x,t)\,dx+V=\rho A(x)\,dx\frac{\partial^2 w}{\partial t^2}(x,t) \tag{8.62}$$

where ρ is the mass density and $A(x)$ is the cross-sectional area of the beam. The moment equation of motion about the y axis passing through the point 0 in Fig. 8.15 leads to

ΣM=0: $$(M+dM)-(V+dV)\,dx+f(x,t)\,dx\frac{dx}{2}-M=0 \tag{8.63}$$

Simplifying

$$-\frac{\partial V}{\partial x}(x,t)+f(x,t)=\rho A(x)\frac{\partial^2 w}{\partial t^2}(x,t) \tag{8.64}$$

$$\frac{\partial M}{\partial x}(x,t)-V(x,t)=0 \tag{8.65}$$

By using the relation $V=\partial M/\partial x$ from Eq. (8.65), Eq. (8.64) becomes

$$-\frac{\partial^2 M}{\partial x^2}(x,t)+f(x,t)=\rho A(x)\frac{\partial^2 w}{\partial t^2}(x,t) \tag{8.66}$$

From the elementary theory of bending of beams (also known as the *Euler-Bernoulli* or *thin beam theory*), the relationship between bending moment and deflection can be expressed as [8.8]

$$M(x,t)=EI(x)\frac{\partial^2 w}{\partial x^2}(x,t) \tag{8.67}$$

Final Equation

where E is Young's modulus and $I(x)$ is the moment of inertia of the beam cross section about the y axis. Inserting Eq. (8.67) into Eq. (8.66), we obtain the equation of motion for the forced lateral vibration of a nonuniform beam:

$$\frac{\partial^2}{\partial x^2}\left[EI(x)\frac{\partial^2 w}{\partial x^2}(x,t)\right]+\rho A(x)\frac{\partial^2 w}{\partial t^2}(x,t)=f(x,t) \tag{8.68}$$

Assume uniform beam (dA/dx=0) and free vibration (f(x,t)=0)

For free vibration, $f(x,t)=0$, and so the equation of motion becomes

$$c^2\frac{\partial^4 w}{\partial x^4}(x,t)+\frac{\partial^2 w}{\partial t^2}(x,t)=0 \tag{8.70}$$

where

$$c=\sqrt{\frac{EI}{\rho A}} \tag{8.71}$$

IMPORTANT: fourth order in space and second order in time
need 4 boundary conditions and 2 initial conditions

Solving Equation

- See example in notes
- Use separation of variables
- Use appropriate BC/ICs
- Turn the crank

Mode Shapes or Eigenvalues

$$W(x) = C_1\left(\cos\beta x + \cosh\beta x\right) + C_2\left(\cos\beta x - \cosh\beta x\right) + C_3\left(\sin\beta x + \sinh\beta x\right) + C_4\left(\sin\beta x - \sinh\beta x\right)$$

Natural Frequencies Eigenvalues

$$\omega = \beta^2\sqrt{\frac{EI}{\rho A}} = \left(\beta l\right)^2\sqrt{\frac{EI}{\rho A l^4}}$$