ME 517: Micro- and Nanoscale Processes

Lecture 21: Microfluidics - Couette Flow I

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Spring 2014

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Boundary Conditions

Generalized slip flow model for ***liquid or gas***:

$$\Delta u\big|_{wall} = u_{fluid} - u_{wall} = L_s \left.\frac{\partial u}{\partial y}\right|_{wall}$$

Slip flow model ***strictly for ideal gases***:

$$u_{gas} - u_{wall} = \lambda \frac{2-\sigma_v}{\sigma_v} \left.\frac{\partial u}{\partial y}\right|_{wall} + \frac{3}{4} \frac{\mu}{\rho T_{gas}} \left.\frac{\partial T}{\partial x}\right|_{wall}$$

Dimensionless Slip flow model for ideal gas:

$$u^*_{gas} - u^*_{wall} = Kn \frac{2-\sigma_v}{\sigma_v} \left.\frac{\partial u^*}{\partial y^*}\right|_{wall} + \frac{3}{2\pi} \frac{\gamma-1}{\gamma} \frac{Kn^2 \operatorname{Re}}{Ec} \left.\frac{\partial T^*}{\partial x^*}\right|_{wall}$$

Navier-Stokes Equation (Dimensionless)

$$\mathrm{Re}\frac{D\vec{V}'}{Dt'} = \mathrm{Re}\left(\frac{\partial \vec{V}'}{\partial t'} + \left(\vec{V}' \cdot \nabla'\right)\vec{V}'\right) = -\nabla' p' + \nabla'^2 \vec{V}'$$

$$\text{where } \mathrm{Re} = \frac{\rho u L}{\mu}$$

- Re=Reynolds number, ratio of inertial forces to viscous forces
- All variables normalized so that they are order unity

Stokes Flows

- Assume we're considering a water flow
  - $\mu = 10^{-3}$ kg/(s m), $\rho = 10^{3}$ kg/m^{3}
- Length scale: 10 μm=10^{-5} m
- Velocity scale: 10 mm/s=10^{-2} m/s
- Then: Re=10^{-1}, so N-S eqn

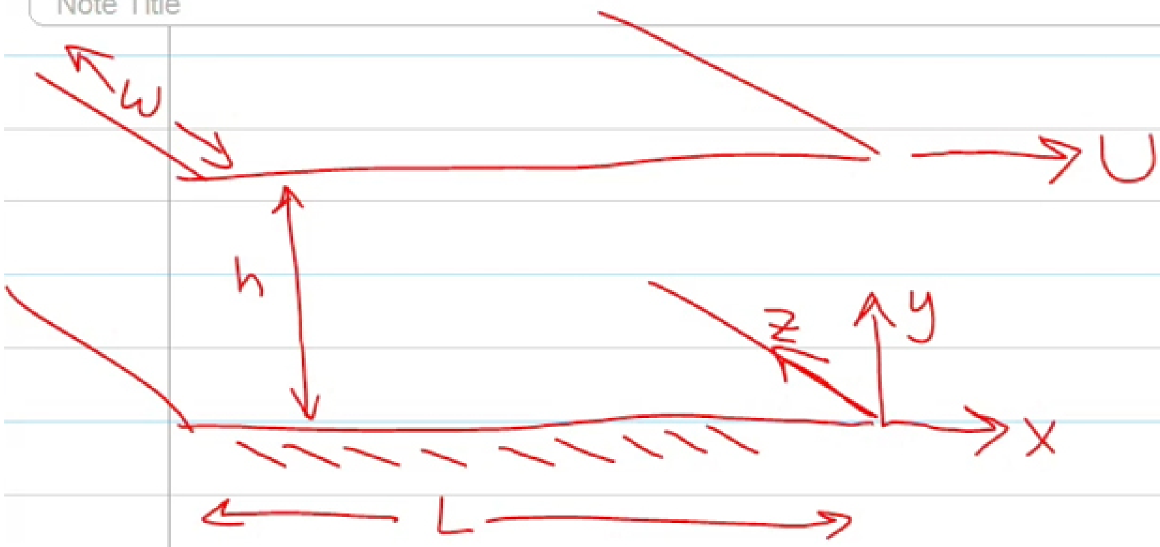
$$0 = -\nabla p + \nabla^2 \vec{V}$$

- becomes Poisson Eqn (Stokes Flow), simple, linear equation

Plane Couette Flow

if $h \ll L, w$, planes ∞

Note Title 3/24/2014



$$\text{NSE}: \rho \frac{D\bar{V}}{Dt} = -\nabla P + \mu \nabla^2 \bar{V}$$

$$\rho\left[\underset{\substack{\text{local}\\ \text{acc}}}{\frac{\partial \bar{V}}{\partial t}} + \underbrace{(\bar{V}\cdot\nabla)\bar{V}}_{\substack{\text{conv}\\ \text{acc}}}\right] = -\nabla P + \mu \nabla^2 \bar{V}$$

$$\left\{\begin{aligned}
\hat{i}:\ & \rho\left[\cancelto{0}{\frac{\partial u}{\partial t}} + u\cancelto{0}{\frac{\partial u}{\partial x}} + v\frac{\partial u}{\partial y} + w\cancelto{0}{\frac{\partial u}{\partial z}}\right] = -\cancelto{0}{\frac{\partial P}{\partial x}} + \mu\left[\cancelto{0}{\frac{\partial^2 u}{\partial x^2}} + \frac{\partial^2 u}{\partial y^2} + \cancelto{0}{\frac{\partial^2 u}{\partial z^2}}\right]\\
\hat{j}:\ & \rho\left[\cancelto{0}{\frac{\partial v}{\partial t}} + u\cancelto{0}{\frac{\partial}{\partial x}} + v\frac{\partial v}{\partial y} + w\cancelto{0}{\frac{\partial v}{\partial z}}\right] = -\frac{\partial P}{\partial y} + \mu\left[\cancelto{0}{\frac{\partial^2 v}{\partial x^2}} + \frac{\partial^2 v}{\partial y^2} + \cancelto{0}{\frac{\partial v}{\partial z^2}}\right]\\
\hat{k}:\ & \rho\left[\cancelto{0}{\frac{\partial w}{\partial t}} + u\cancelto{0}{\frac{\partial}{\partial x}} + v\frac{\partial w}{\partial y} + w\cancelto{0}{\frac{\partial w}{\partial z}}\right] = -\cancelto{0}{\frac{\partial P}{\partial z}} + \mu\left[\cancelto{0}{\frac{\partial^2 w}{\partial x^2}} + \frac{\partial^2 w}{\partial y^2} + \cancelto{0}{\frac{\partial^2 w}{\partial z^2}}\right]
\end{aligned}\right.$$

$$\nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

$$\bar{V} = u\hat{i} + v\hat{j} + w\hat{k}$$

$$= v_x\hat{i} + v_y\hat{j} + v_z\hat{k}$$

$$\nabla P = \hat{i}\frac{\partial P}{\partial x} + \hat{j}\frac{\partial P}{\partial y} + \hat{k}\frac{\partial P}{\partial z}$$

$$\nabla \cdot \bar{V} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot \left(u\hat{i} + v\hat{j} + w\hat{k}\right)$$

$$= \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

$$(\bar{V} \cdot \nabla)\bar{V} = \left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}\right)\left(u\hat{i} + v\hat{j} + w\hat{k}\right)$$

$$= \hat{i}\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) + \hat{j}\left(\qquad\right) + \hat{k}\left(\qquad\right)$$

$$\nabla^2 \bar{V} = (\nabla \cdot \nabla)\bar{V} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\bar{V}$$

$$= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\left(\hat{i}u + \hat{j}v + \hat{k}w\right)$$