ME 517: Micro- and Nanoscale Processes

Lecture 22: Microfluidics - Couette Flow II

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Plane Couette Flow

if $h \ll L, w$, planes ∞

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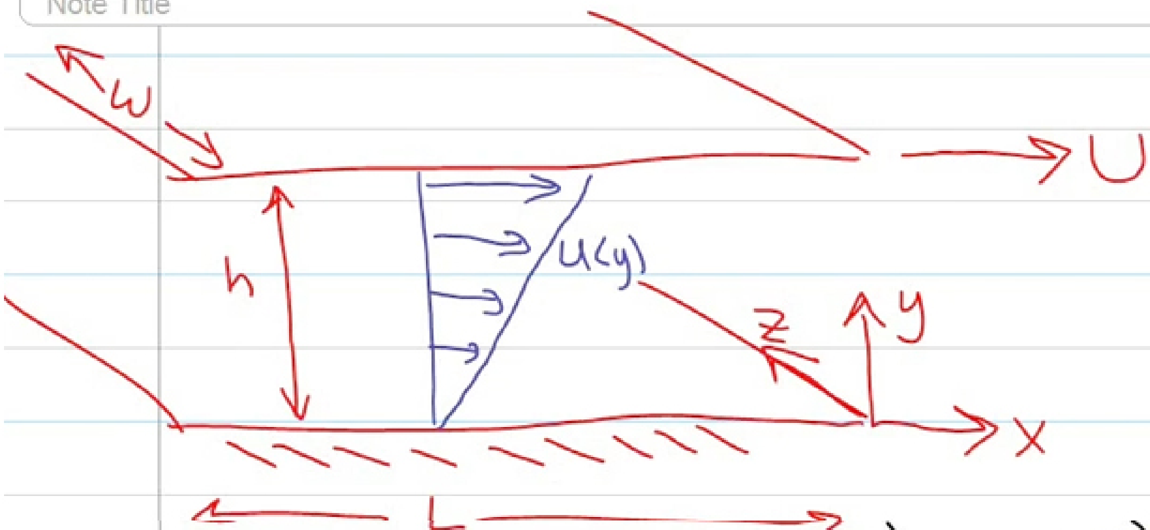


translational symm

$x: \frac{\partial(\,)}{\partial x} = 0$

$w: \frac{\partial(\,)}{\partial z} = 0$

Steady state $\frac{\partial(\,)}{\partial t} = 0$

COM: $\nabla \cdot \bar{V} = 0$ (for incomp fluid)

$$\boxed{\cancel{\frac{\partial u}{\partial x}}^{\,0} + \frac{\partial v}{\partial y} + \cancel{\frac{\partial w}{\partial z}}^{\,0} = 0} \Rightarrow \frac{\partial v}{\partial y} = 0$$

$v = C_1$

$$\rho\left[\underbrace{\frac{\partial \bar V}{\partial t}}_{\text{local acc}} + \underbrace{(\bar V\cdot\nabla)\bar V}_{\text{conv acc}}\right] = -\nabla P + \mu\nabla^2\bar V$$

$$\hat i:\ \rho\left[\cancelto{0}{\frac{\partial u}{\partial t}} + \cancelto{0}{u\frac{\partial u}{\partial x}} + \cancelto{0}{v\frac{\partial u}{\partial y}} + \cancelto{0}{w\frac{\partial u}{\partial z}}\right] = -\cancelto{0}{\frac{\partial P}{\partial x}} + \mu\left[\cancelto{0}{\frac{\partial^2 u}{\partial x^2}} + \frac{\partial^2 u}{\partial y^2} + \cancelto{0}{\frac{\partial^2 u}{\partial z^2}}\right]$$

$$\hat j:\ \rho\left[\cancelto{0}{\frac{\partial v}{\partial t}} + \cancelto{0}{u\frac{\partial}{\partial x}} + \cancelto{0}{v\frac{\partial v}{\partial y}} + \cancelto{0}{w\frac{\partial v}{\partial z}}\right] = -\frac{\partial P}{\partial y} + \mu\left[\cancelto{0}{\frac{\partial^2 v}{\partial x^2}} + \cancelto{0}{\frac{\partial^2 v}{\partial y^2}} + \cancelto{0}{\frac{\partial v}{\partial z^2}}\right]$$

$$\hat k:\ \rho\left[\cancelto{0}{\frac{\partial w}{\partial t}} + \cancelto{0}{u\frac{\partial}{\partial x}} + \cancelto{0}{v\frac{\partial w}{\partial y}} + \cancelto{0}{w\frac{\partial w}{\partial z}}\right] = -\cancelto{0}{\frac{\partial P}{\partial z}} + \mu\left[\cancelto{0}{\frac{\partial^2 w}{\partial x^2}} + \frac{\partial^2 w}{\partial y^2} + \cancelto{0}{\frac{\partial^2 w}{\partial z^2}}\right]$$

$$\rho\left[\underset{\text{local acc}}{\frac{\partial \bar{V}}{\partial t}} + \underbrace{(\bar{V}\cdot\nabla)\bar{V}}_{\text{conv acc}}\right] = -\nabla P + \mu\nabla^2\bar{V}$$

$$\hat{i}: \rho\left[\cancelto{0}{\frac{\partial u}{\partial t}} + \cancelto{0}{u\frac{\partial u}{\partial x}} + \cancelto{0}{v\frac{\partial u}{\partial y}} + \cancelto{0}{w\frac{\partial u}{\partial z}}\right] = -\cancelto{0}{\frac{\partial P}{\partial x}} + \mu\left[\cancelto{0}{\frac{\partial^2 u}{\partial x^2}} + \frac{\partial^2 u}{\partial y^2} + \cancelto{0}{\frac{\partial^2 u}{\partial z^2}}\right]$$

$$\hat{j}: \rho\left[\cancelto{0}{\frac{\partial v}{\partial t}} + \cancelto{0}{u\frac{\partial}{\partial x}} + \cancelto{0}{v\frac{\partial v}{\partial y}} + \cancelto{0}{w\frac{\partial v}{\partial z}}\right] = -\frac{\partial P}{\partial y} + \mu\left[\cancelto{0}{\frac{\partial^2 v}{\partial x^2}} + \cancelto{0}{\frac{\partial^2 v}{\partial y^2}} + \cancelto{0}{\frac{\partial^2 v}{\partial z^2}}\right]$$

$$\Rightarrow P = P_0$$

$$\hat{k}: \rho\left[\cancelto{0}{\frac{\partial w}{\partial t}} + \cancelto{0}{u\frac{\partial}{\partial x}} + \cancelto{0}{v\frac{\partial w}{\partial y}} + \cancelto{0}{w\frac{\partial w}{\partial z}}\right] = -\cancelto{0}{\frac{\partial P}{\partial z}} + \mu\left[\cancelto{0}{\frac{\partial^2 w}{\partial x^2}} + \frac{\partial^2 w}{\partial y^2} + \cancelto{0}{\frac{\partial^2 w}{\partial z^2}}\right]$$

- Parallel flows with non-porous boundaries, same as Stokes flow
- $Re \ll 1 \Rightarrow$ Stokes Eqn, low Re # flow

When $Re \gg 1$, turbulence is important
($Re \approx 2300$)

$Re \gtrsim 1 \rightarrow$ inertial effects, vortices

$x: \quad \dfrac{d^2 u}{dy^2} = 0 \Rightarrow u(y) = C_2 y + C_3 \qquad u(0) = 0,\ u(h) = U$

$z: \quad \dfrac{d^2 w}{dy^2} = 0 \Rightarrow w(y) = C_4 y + C_5 \qquad w(0) = 0,\ w(h) = 0$

$$\boxed{w(y) = 0}$$

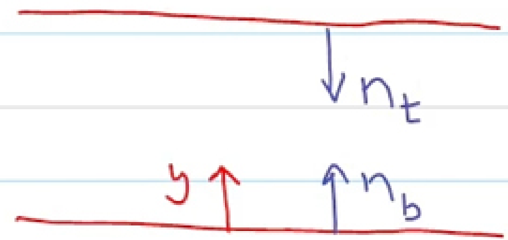
$\Rightarrow \quad U = C_2 h + C_3 \Rightarrow C_2 = U/h$

$\quad\quad 0 = C_2 \times 0 + C_3 \Rightarrow C_3 = 0$

$\Rightarrow \boxed{u(y) = \dfrac{Uy}{h}}$ BCs check out

Consider Navier's slip BC now

$$\Delta u = u_f - u_w = L_s \frac{\partial u}{\partial n}\Big|_{n \to \text{wall}}$$



$$\hat{n}_b = \hat{j} \; ; \; \hat{n}_t = -\hat{j}$$

bot: $\Delta u = u_f - \underbrace{\cancel{u_w}}_{\to 0} = L_s \frac{\partial u}{\partial y}\Big|_{y \to 0}$

top: $\Delta u = u_f - U = L_s (-1) \frac{\partial u}{\partial y}\Big|_{y \to h}$