ME 517: Micro- and Nanoscale Processes

Lecture 25: Microfluidics - Stokes Sphere Drag I

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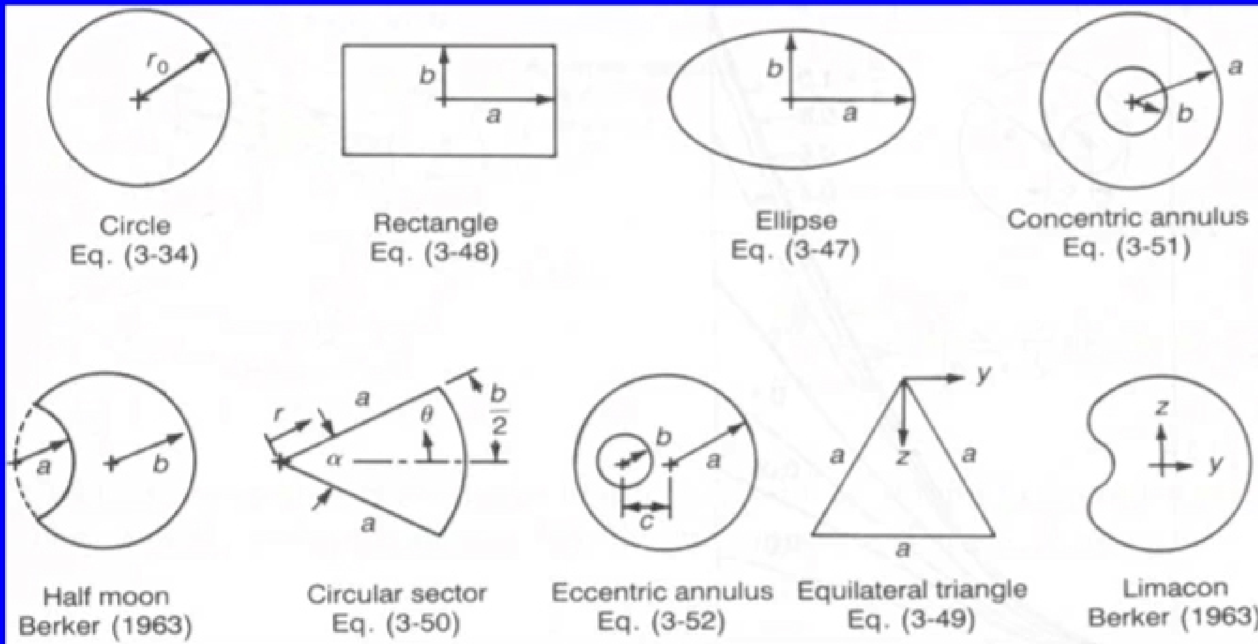
Continuum incompressible flow through long channels (White, 1991)



$$u = \frac{-d\hat{p}/dx}{4\mu}\left(r_0^2 - r^2\right)$$

$$Q_{\text{pipe}} = \frac{\pi r_0^4}{8\mu}\left(-\frac{d\hat{p}}{dx}\right)$$

$$u(y,z) = \frac{16a^2}{\mu\pi^3}\left(-\frac{d\hat{p}}{dx}\right)\sum_{i=1,3,5,\ldots}^{\infty}(-1)^{(i-1)/2}\left[1 - \frac{\cosh(i\pi z/2a)}{\cosh(i\pi b/2a)}\right] \times \frac{\cos(i\pi y/2a)}{i^3}$$

$$Q = \frac{4ba^3}{3\mu}\left(-\frac{d\hat{p}}{dx}\right)\left[1 - \frac{192a}{\pi^5 b}\sum_{i=1,3,5,\ldots}^{\infty}\frac{\tanh(i\pi b/2a)}{i^5}\right]$$

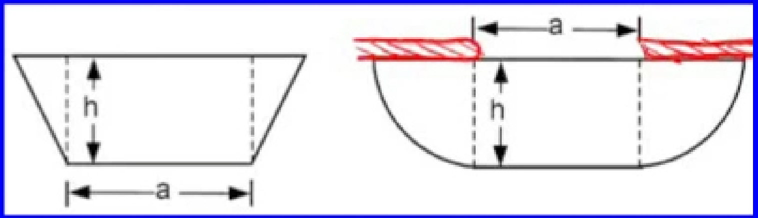
$$u(y,z) = \frac{-d\hat{p}/dx}{2\sqrt{3}\,a\mu}\left(z - \frac{1}{2}a\sqrt{3}\right)\left(3y^2 - z^2\right)$$

$$Q = \frac{a^4\sqrt{3}}{320\mu}\left(-\frac{d\hat{p}}{dx}\right)$$

Micro/Nanoscale Physical Processes

Shapes *NOT* represented in table

- Two additional useful shapes:
  - Trapezoid
  - Rectangle with two rounded corners
    - I have not found any references that provide these solutions—should be easy paper in microfluidics

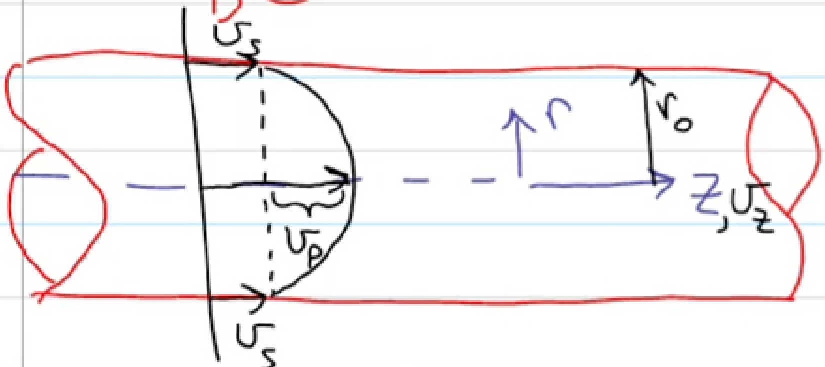


- For shapes in which we don't have analytical solution
  - Approximate approach
  - Hydraulic diameter (4*Area/Perimeter)
  - Plug into circle expression for Q versus ΔP

$\frac{1}{4} < \frac{h}{a} < 4$

10%

TUBE



$$v_z = v_p\left[1 - \left(\frac{r}{r_0}\right)^2\right] + v_s$$

$$\left.v_p = -\frac{r_0^2}{4\mu}\frac{dP}{dz}\right\}\text{Poiseuille}$$

$$Q = \pi r_0^2\left(\frac{1}{2}v_p + v_s\right)$$

Stokes Flow Small Spheres

$$0 = -\nabla P + \nabla^2 \bar{V}$$ Stokes Flow

Take the divergence

$$0 = -\nabla \cdot (\nabla P) + \nabla \cdot (\nabla^2 \bar{V})$$

$$= -\nabla^2 P + \nabla^2 (\cancel{\nabla \cdot \bar{V}})$$ incomp flow

$\nabla \cdot \bar{V} \to 0$

$$\boxed{0 = \nabla^2 P}$$

Pressure and Velocity separable in Stokes flows

Stokes Flow Small Spheres

$$0 = -\nabla P + \nabla^2 \bar{V}$$ Stokes Flow

Take the divergence

$$0 = -\nabla \cdot (\nabla P) + \nabla \cdot (\nabla^2 \bar{V})$$

$$= -\nabla^2 P + \nabla^2 (\cancel{\nabla \cdot \bar{V}}) \quad 0$$ incomp flow

$$\boxed{0 = \nabla^2 P}$$ ← Pressure and Velocity separable in Stokes flows

Take curl of Stokes Eqn

$$0 = -\cancel{\nabla \times (\nabla P)} + \nabla \times (\nabla^2 \bar{V})$$

0 identity

When we have 2D, planar flow, can simplify further

Define stream function $\psi(x,y)$ such that

$$u = \frac{\partial \psi}{\partial y} \; ; \; v = -\frac{\partial \psi}{\partial x}$$

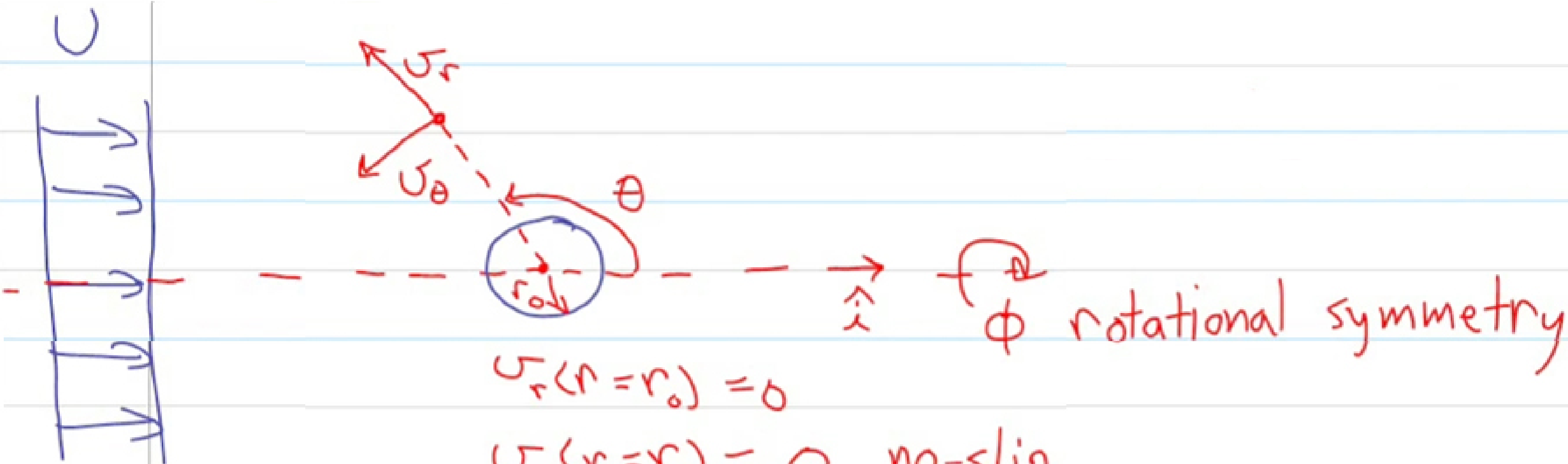
significance is if I identify $\psi = \text{const}$ curves; those are streamlines.

$$\nabla \times \bar{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & 0 \\ u & v & 0 \end{vmatrix} = \hat{k}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = -\hat{k}\left(+\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}\right) = -\hat{k}\,\nabla^2 \psi$$

$$\nabla^2(\nabla \times \bar{V}) = \nabla^2(\nabla^2 \psi) = \boxed{\nabla^4 \psi = 0}$$

Stokes flow
2D planar or axisymmetri

biharmonic operator
spherical polar coords



$v_r(r=r_0)=0$

$v_\theta(r=r_0)=0$ no-slip

$\neq 0$ slip flow

BC as $r\to\infty$