ME 517: Micro- and Nanoscale Processes

Lecture 26: Microfluidics - Stokes Sphere Drag II

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Spring 2014

Stokes Flow Small Spheres

$$0 = -\nabla P + \nabla^2 \bar{V}$$ Stokes Flow

Take the divergence

$$0 = -\nabla\cdot(\nabla P) + \nabla\cdot(\nabla^2 \bar{V})$$

$$= -\nabla^2 P + \nabla^2 \cancel{(\nabla\cdot\bar{V})}_{0}$$ incomp flow

$$\boxed{0 = \nabla^2 P}$$

Pressure and Velocity separable in Stokes flows

Stokes Flow Small Spheres

$$0 = -\nabla P + \nabla^2 \bar{V}$$ Stokes Flow

Take the divergence

$$0 = -\nabla \cdot (\nabla P) + \nabla \cdot (\nabla^2 \bar{V})$$

$$= -\nabla^2 P + \nabla^2 (\cancel{\nabla \cdot \bar{V}})$$ incomp flow

$\cancel{\nabla \cdot \bar{V}} \to 0$

$$\boxed{0 = \nabla^2 P}$$ $\leftarrow$ Pressure and Velocity separable in Stokes flows

Take curl of Stokes Eqn

$$0 = -\cancel{\nabla \times (\nabla P)} + \nabla \times (\nabla^2 \bar{V})$$

$\to 0$ identity

When we have 2D, planar flow, can simplify further

Define stream function $\psi(x,y)$ such that

$$u = \frac{\partial \psi}{\partial y} \; ; \; v = -\frac{\partial \psi}{\partial x}$$

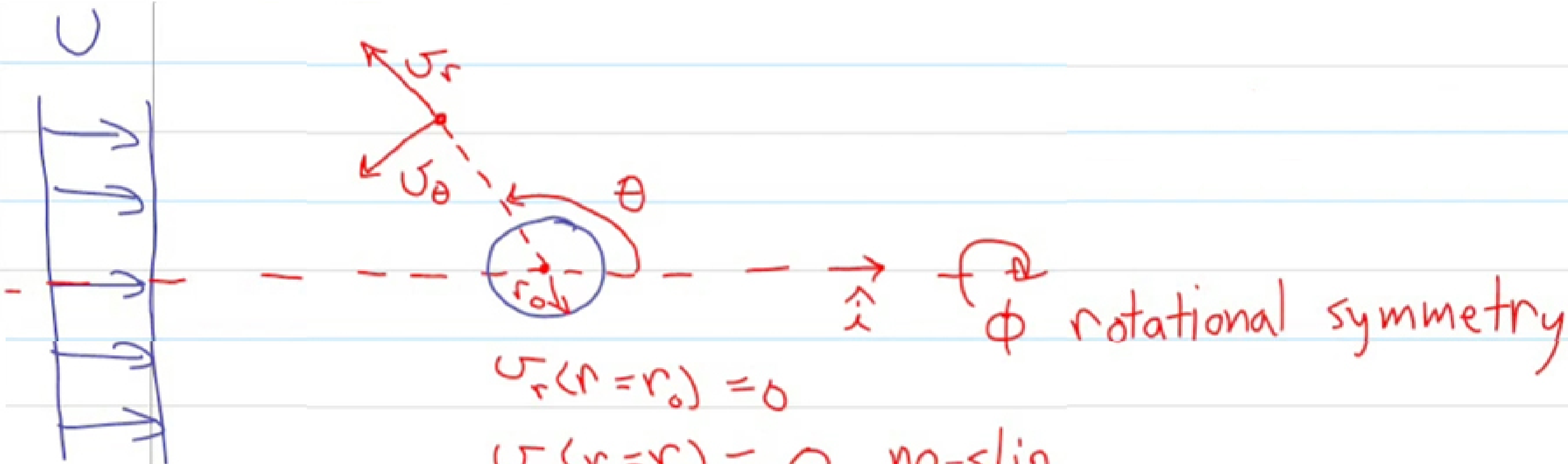
significance is if I identify $\psi = \text{const}$ curves; those are streamlines.

$$\nabla \times \bar{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & 0 \\ u & v & 0 \end{vmatrix} = \hat{k}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = -\hat{k}\left(+\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}\right) = -\hat{k}\,\nabla^2 \psi$$

$$\nabla^2(\nabla \times \bar{V}) = \nabla^2(\nabla^2 \psi) = \boxed{\nabla^4 \psi = 0}$$

Stokes flow
2D planar or axisymmetri

biharmonic operator
spherical polar coords



$v_r(r=r_0)=0$

$v_\theta(r=r_0)=0$ no-slip

$\neq 0$ slip flow

BC as $r\to\infty$

Panton 3rd ed

$$\nabla^4\psi = \nabla^2\nabla^2\psi = \left[\frac{\partial^2}{\partial r} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right)\right]^2\psi_{(r,\theta)} = 0$$

$$U_r = \frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta} \; ; \; U_\theta = -\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r}$$

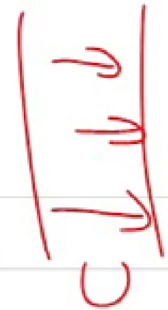
as $r\to\infty$; $Ur^2\sin\theta\cos\theta = \frac{\partial\psi}{\partial\theta}$ (similar exp $\frac{\partial\psi}{\partial r}$)

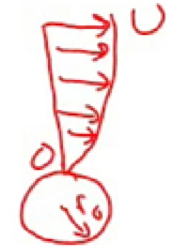
on $r=r_o$ $\frac{\partial\psi}{\partial\theta}=0$ and $\frac{\partial\psi}{\partial r}=0$

separate $\psi(r,\theta)$ as $f(r)g(\theta)=\psi(r,\theta)$

much math... $\psi(r,\theta) = \frac{1}{4}U^2 r_o^2\sin^2\theta\left[\frac{r_o}{r} - \frac{3r}{r_o} + 2\frac{r^2}{r_o^2}\right]$

$$u_r = U\cos\theta\left[1+\frac{r_0^2}{2r^3}-\frac{3r_0}{2r}\right]$$





$$u_\theta = U\sin\theta\left[-1+\frac{r_0^3}{4r^3}+\frac{3r_0}{4r}\right]$$

u_r, u_θ always smaller than U

$$F_D = \underbrace{4\pi\mu U r_0}_{\tau} + \underbrace{2\pi\mu U r_0}_{P} = 6\pi\mu U r_0 \quad \text{(no slip)}$$

What if we allow slip @ sphere's surface?

in Cartesian coords $\Delta u = u_{slip} = L_s \frac{\partial u}{\partial y}\Big|_{y\to 0}$ Navier 1836

$$= L_s \frac{1}{\mu}\tau_{yx}\Big|_{y\to 0}$$

$$= L_s \frac{1}{\mu} \tau_{y*}\Big|_{y\to 0}$$

in sp-pol coords

$$U_\theta (r \to r_0) = \frac{L_s}{\mu} \tau_{r\theta} (r \to r_0)$$

in ψ form

$$\left.\frac{\partial \psi}{\partial r}\right|_{r\to r_0} = L_s \frac{\partial}{\partial r}\left(\frac{1}{r^2} \frac{\partial \psi}{\partial r}\right)\Big|_{r\to r_0} \quad \Big\} \text{ changed BCs}$$

turn the crank

$$F_D = 6\pi \mu U r_0 \frac{1+2L_s}{1+3L_s}$$