ECE695: Reliability Physics of Nano-Transistors
Lecture 34A: Appendix
- Variability by Bootstrap Method

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Appendix: Variability by Bootstrap method

Ref. courses.washington.edu/matlab2/Lesson_6.html
Uncertainty in parameters: Least Square

Is the error in \( W \) Gaussian distributed?

\[
W \equiv \beta \ln t + c \quad \ln t \equiv \beta^{-1}W - \beta^{-1}c = a^*W + b^*
\]

Inverse fitting is more appropriate … \( x = a^* + b^* y \)

\[
a^* = \frac{n \sum x_i y_i - \sum x_i \sum y_i}{n \sum y_i^2 - (\sum y_i)^2}
\]

\[
b^* = \frac{n \sum x_i y_i - \sum x_i \sum y_i}{n \sum x_i^2 - (\sum x_i)^2}
\]

\[
\sigma_{\beta}^2 = \left( \frac{\delta \beta}{\delta a} \right)^2 \sigma_a^2 + \left( \frac{\delta \beta}{\delta b} \right)^2 \sigma_b^2
\]

\[
\sigma_c^2 = \left( \frac{\delta \beta}{\delta a} \right)^2 \sigma_a^2 + \left( \frac{\delta \beta}{\delta b} \right)^2 \sigma_b^2
\]
MLE estimator for Weibull: Did not discuss variability

Recall

\[ f(t; \alpha, \beta) = \frac{\beta}{\alpha^\beta} \cdot t^{\beta-1} \cdot e^{-\left(\frac{t}{\alpha}\right)^\beta} \]

\[
\ln L = \sum_{i=1}^{n} \ln f(t_i, \alpha, \beta) \\
= n \ln \beta - n \ln \alpha + (\beta - 1) \sum_{i=1}^{n} \ln t_i / \alpha - \sum_{i=1}^{n} \left(\frac{t_i}{\alpha}\right)^\beta
\]

\[
\frac{d \ln L}{d \alpha} = 0 \quad \frac{d \ln L}{d \beta} = 0
\]

\[
\left( \sum_{i=1}^{n} t_i^\alpha \ln(t_i)^\beta \middle/ \sum_{i=1}^{n} t_i^\beta \right) - \frac{1}{n} \sum_{i=1}^{n} \ln(t_i)^\beta = 1
\]

\[
\alpha = \left[ \frac{1}{n} \sum_{i=1}^{n} t_i^\beta \right]^{\frac{1}{\beta}}
\]

Solve for unknowns \( \alpha, \beta \)
(1) Bootstrap method: Introduction

- Bootstrap method: Introduction

- 68% between +0.2 to -0.2
- 95% between +0.4 to -0.4

\[ s = \sqrt{\frac{\sum_{j=1}^{N=25} (t_j - \langle t \rangle)^2}{N - 1}} \sim \sqrt{\frac{1}{24}} \sim 0.2 \]

Ref. courses.washington.edu/matlab2/Lesson_6.html
Working with a single sample

All you have is a single sample..

Generate synthetic samples from the original (with replacement)

- Synthetic sample 1:
  0.2  -0.1  -0.6  -0.1  0.5

- Synthetic sample 2:
  0.3  0.2  -0.6  0.2  0.5

- Synthetic sample 3:
  0.5  -0.1  0.5  0.2  0.3
Multiple sample vs. single sample

Bootstrap average is not zero!
And yet, the $s \sim 0.18$, just from a single sample.
The success of the method relies on precision measurement.
Parametric vs. non-parametric
Bootstrap

0.2 -0.1 0.5 0.3 -0.6  
Fit the distribution of your choice by
Maximum likelihood estimators (MLE)
(obtain parameters, i.e. $\eta_0, \beta_0$)

Generate synthetic samples based on the parametric distribution

0.12 -0.17 -0.44 -0.71 0.52  
Synthetic sample 1 (new $\eta_1, \beta_1$)

0.32 0.21 -0.69 0.23 0.58  
Synthetic sample 2 (new $\eta_2, \beta_2$)

Plot distribution of statics $\eta_i, \beta_i$
Distribution of $\alpha$ and $\beta$

Distribution of $\eta$

PDF

Distribution of $\beta$

PDF

Same technique for polling and tenure rate of faculty!
Why resampling from the same distribution generates new fit parameters

Samples taken from the same distribution \((\eta_0, \beta_0)\) generates datapoints that are fitted with new \((\eta_i, \beta_i)\)
# References

