Optical Cavities

Cavity provides the feedback system needed for laser operation!

A cavity mode is a field distribution that reproduces itself in relative shape and in relative phase after a round trip.

To follow a Gaussian beam through a round trip in a cavity we employ the ABCD law.

1. We assume that the Hermite-Gaussian beams are the characteristic modes of the optical cavity (Mirrors exactly match the surfaces of constant phase of the beams.)

For this require that:
2. The complex beam parameter repeats itself after a round trip!

\[ q(z_1 + \text{roundtrip}) = q(z_1) \]
\[ q(z_1 + \text{roundtrip}) = q(z_1) \]

\[ q(z_1) = \frac{Aq(z_1) + B}{Cq(z_1) + D} \]

\[ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \] transmission matrix for the unit cell (round trip)

\[ Cq^2 + Dq = Aq + B \]

\[ B \left( \frac{1}{q} \right)^2 + 2 \left( \frac{A - D}{2} \right) \left( \frac{1}{q} \right) - C = 0 \]

\[ \frac{1}{q} = -\frac{A - D}{2B} \pm \frac{1}{B} \left[ \left( \frac{A - D}{2} \right)^2 + BC \right]^{\frac{1}{2}} \]

\[ AD - BC = 1 \]

\[ \frac{A^2}{4} - \frac{AD}{2} + \frac{D^2}{4} + BC \left( 1 - \frac{AD}{2} + \frac{AD}{2} \right) = \left( \frac{A + D}{2} \right)^2 - 1 \]

\[ \frac{1}{q} = -\frac{A - D}{2B} - j \left[ 1 - \left( \frac{A + D}{2} \right)^2 \right]^{\frac{1}{2}} \]
\[
\frac{1}{q} = \frac{1}{R} - j \frac{\lambda}{\pi w^2}
\]

\[R(z_1) = -\frac{2B}{A-D}\]

\[\frac{\pi w^2(z)}{\lambda} = \frac{B}{\left[1-\left(\frac{A+D}{2}\right)^2\right]^{\frac{1}{2}}}
\]

\[
T = \begin{pmatrix}
1 & d + z_1 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & d - z_1 \\
-\frac{1}{f} & 1 - \frac{d - z_1}{f}
\end{pmatrix}
\]

If we start the unit cell at the flat mirror \((z=0)\)

\[
T = \begin{pmatrix}
1 & d \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & d \\
-\frac{1}{f} & 1 - \frac{d}{f}
\end{pmatrix}
= \begin{pmatrix}
1 - \frac{d}{f} & d + d\left(1 - \frac{d}{f}\right) \\
0 & 1 - \frac{d}{f}
\end{pmatrix}
\]

\[
\frac{\pi w_0^2}{\lambda} = \frac{d + d\left(1 - \frac{d}{f}\right)}{\left[1 - \left(1 - \frac{d}{f}\right)^2\right]^{\frac{1}{2}}}
= \frac{2d\left(1 - \frac{d}{R}\right)}{\left[4\left(\frac{d}{R}\right)\left(1 - \frac{d}{R}\right)\right]^{\frac{1}{2}}}
= \frac{f = \frac{R}{2}}
\]

\[
\frac{\pi w_0^2}{\lambda} = (dR)^{\frac{1}{2}}\left(1 - \frac{d}{R}\right)^{\frac{1}{2}}
\]
$$w^2(z) = w_0^2 \left[ 1 + \left( \frac{z}{z_0} \right)^2 \right]$$

$$z_0^2 = \left( \frac{\pi w_0^2}{\lambda} \right)^2$$

$$\frac{\pi w^2(d)}{\lambda} = \left( dR \right)^{\frac{1}{2}} \left( 1 - \frac{d}{R} \right)^{\frac{1}{2}} \left[ 1 + \frac{d^2}{dR \left( 1 - \frac{d}{R} \right)} \right] = \frac{(dR)^{\frac{1}{2}}}{\left( 1 - \frac{d}{R} \right)^{\frac{1}{2}}}$$

On a spherical mirror

Gaussian beam analysis does not apply

(not physical)

$$\frac{\lambda \sqrt{dR_2}}{\pi}$$

$$\frac{\pi w_0^2}{\lambda} = \left( dR \right)^{\frac{1}{2}} \left( 1 - \frac{d}{R} \right)^{\frac{1}{2}}$$

becomes imaginary
The requirement that mirrors must match the surface of the constant phase (so that a cavity mode is excited) allows one to find where the mirrors with $R_1$ & $R_2$ should be placed, in general case.

$\text{R}_1, \text{R}_2$ are given, we look for $z_1 z_2$

1) $z_1 + z_2 = d$
2) $R(z_2) = R_2 = z_2 \left[1 + \left(\frac{z_0}{z_2}\right)^2\right]$
3) $R(z_1) = -R_1 = -z_1 \left[1 + \left(\frac{z_0}{z_1}\right)^2\right]$

The wave front on the left at $z=0$ has a (mathematically) negative radius of curvature, but we know that the mirror $R_1$ has positive (focusing) properties. We treat all distances $z_1, z_2$ as positive numbers and let the radii of curvature of the mirrors carry their own sign.

Solving (involving algebra)....

$$z_0^2 = \left(\frac{\pi w_0^2}{\lambda}\right)^2 = \frac{d (R_1 - d) (R_2 - d) (R_1 + R_2 - d)}{(R_1 + R_2 - 2d)^2}$$

$$z_1 = \frac{d (R_2 - d)}{R_1 + R_2 - 2d} \quad z_2 = \frac{d (R_1 - d)}{R_1 + R_2 - 2d}$$

$R_1 = \infty \quad z_0^2 = \pi w_0^2 \lambda \Rightarrow dR_2 \left(1 - \frac{d}{R_2}\right) \quad z_1 = 0; \ z_2 = d \quad \frac{\pi w_0^2}{\lambda} = (dR)^{\frac{1}{2}} \left(1 - \frac{d}{R}\right)^{\frac{1}{2}}$

Same result
Summary

1) Postulate that Hermite-Gaussian Beams are the normal modes for the cavity.

2) Formulate an equivalent transmission system for this cavity showing one round trip. Identify a unit cell.

3) Force the complex beam parameter to transform into itself after a round trip by use of the ABCD law.

4) Evaluate $R$ and $w$ using:

$$R(z) = -\frac{2B}{A - D}$$

$$\frac{\pi w^2(z)}{\lambda} = \frac{B}{\left[1 - \left(\frac{A + D}{2}\right)^2\right]^{\frac{1}{2}}}$$

The theory applies for Stable cavity only!
Mode Volume

\[ E_0^2V = \int_{0}^{d} dz \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy E(x, y, z) E^*(x, y, z) \]

\[ E_0^2V_{m,n} = E_0^2 \int_{0}^{\infty} \frac{w_0^2}{w^2(z)} \, dz \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \, H_m^2\left(\frac{\sqrt{2}x}{w}\right) e^{-\frac{2x^2}{w^2}} \times H_n^2\left(\frac{\sqrt{2}y}{w}\right) e^{-\frac{2y^2}{w^2}} \]

use \( u = \frac{\sqrt{2}x}{w} \) or \( \frac{\sqrt{2}y}{w} \)

\[ V_{m,n} = \int_{0}^{d} \frac{w_0^2}{2} \, dz \int_{-\infty}^{\infty} H_m^2(u) e^{-u^2} \, du \int_{-\infty}^{\infty} H_n^2(u) e^{-u^2} \, du \]

\[ V_{m,n} = \frac{\pi w_0^2}{2} d \left( m! n! 2^{m+n} \right) \]

Area x length  Modification for high-order modes
Example (textbook)

\[ w_0 = 0.94 \text{mm} \quad R_2 = 20 \text{ m} \]
\[ d = 1 \text{ m} \quad R_1 = \infty \]

\[ \rightarrow V_{0,0} = 1.38 \text{cm}^3 \]
\[ = \frac{\pi w_0^2}{2} \times d \]

He-Ne laser
\[ P = 0.1 \text{ torr} \quad \text{(of neon)} \]

Each atom is excited (by the gas discharge) and thus producing a photon at 632.8 µm, say, 10 times per second.

Energy per photon
\[ h \nu = \frac{hc}{\lambda} = 3.14 \times 10^{-19} J = 1.96 eV \]
\[ \times \text{# of Ne atoms} = 0.1(3.54 \times 10^{16})V_{0,0} = 4.88 \times 10^{15} \]
\[ \times \text{(average excitation per atom = average emission per atom)} = 10 \text{ sec}^{-1} \]
\[ = \text{Power} = 15.3 mW \quad \text{typical!} \]