Recap: inferences about quality

1. Confidence Interval
   \[ \bar{X} - Z_{\alpha/2} \frac{\sigma}{\sqrt{n}} \leq \mu \leq \bar{X} + Z_{\alpha/2} \frac{\sigma}{\sqrt{n}} \]

2. Sampling Distribution
   \[ Z_0 = \frac{\bar{X} - \mu}{\sigma / \sqrt{n}} \]

3. Hypothesis Testing
   \[ \bar{X} - \mu \sim \mathcal{N}(0,1) \]
   Reject \( H_0 : \mu = \mu_0 \) if \( \left| \frac{\bar{X} - \mu}{\sigma / \sqrt{n}} \right| > Z_{\alpha/2} \)

4. P-value
   \[ P = 2[1 - \Phi(|Z_0|)] \]
   with two-sided \( H_1 : \mu \neq \mu_0 \)

Reject \( H_0 \) is \( P < \alpha \)
Hypothesis test: type I and type II errors

### Decision based on samples

<table>
<thead>
<tr>
<th>Reality</th>
<th>Reject $H_0$</th>
<th>NOT Reject $H_0$</th>
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<tbody>
<tr>
<td>$H_0$ is True</td>
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<td>Z_0</td>
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<tr>
<td>$\mu = \mu_0$</td>
<td>Type I error ($\alpha$)</td>
<td>Confidence ($1 - \alpha$)</td>
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<tr>
<td>$H_0$ is NOT True</td>
<td>Power ($1 - \beta$)</td>
<td>Type II error ($\beta$)</td>
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<tr>
<td>$\mu \neq \mu_0$</td>
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Reality

- **Reality**: The true situation.$\text{0}_0$ is the population mean.
- **Null Hypothesis ($H_0$)**: $\mu = \mu_0$.
- **Alternative Hypothesis ($H_1$)**: $\mu \neq \mu_0$.

**Decision Rule**:

- **Reject $H_0$** if $|Z_0| > Z_{\alpha/2}$.
- **Do not reject $H_0$** if $|Z_0| < Z_{\alpha/2}$.

**Type I Error ($\alpha$)**: $\mu = \mu_0$ but $H_0$ is rejected.

**Type II Error ($\beta$)**: $\mu \neq \mu_0$ but $H_0$ is not rejected.

**Power ($1 - \beta$)**: The probability of rejecting $H_0$ when $H_0$ is false.

**Confidence ($1 - \alpha$)**: The probability of not rejecting $H_0$ when $H_0$ is true.

**Rejection region**

- Lower Bound ($L$): $\mu_0 - Z_{\alpha/2} \sigma / \sqrt{n}$
- Upper Bound ($U$): $\mu_0 + Z_{\alpha/2} \sigma / \sqrt{n}$

**Decision based on samples**

- **Rejection region** for $H_0$ is from $L$ to $U$.
- **Rejection region** for $H_1$ is outside the $L$ to $U$ range.
Type I and type II errors

• Type I error (false reject/false alarm/false positive):
  – \( \alpha = \) Type I error rate = \( \Pr\{\text{type I error}\} \)
    = \( \Pr\{\text{decide to reject } H_0 \mid H_0 \text{ is true in reality}\} \)
    = \( \Pr\{\text{test statistic falling in the rejection region} \mid H_0 \text{ is true}\} \)

• Type II error (misdetection/false negative):
  – \( \beta = \) Type II error rate = \( \Pr\{\text{type II error}\} \)
    = \( \Pr\{\text{fail to reject } H_0 \mid H_0 \text{ is false}\} \)
    = \( \Pr\{\text{test statistic NOT falling in the rejection region} \mid H_0 \text{ is NOT true}\} \)

• Power of the test (correct detection):
  – Power = \( 1 - \beta = \Pr\{\text{reject } H_0 \mid H_0 \text{ is NOT true}\} \)

Note:
The rejection limits (L & U) are based on \( H_0 \) parameter and the user-specified \( \alpha \).
Type II Error: test mean of a normal population with known $\sigma$

$H_0 : \mu = \mu_0$

$H_1 : \mu \neq \mu_0$

$H_1 : \mu = \mu_1 = \mu_0 + \delta; (\delta \neq 0)$

$\beta = P\{\text{test statistic NOT falling in rejection region} \mid H_0 \text{ is NOT true}\}$

$\beta$ can only be calculated for a given $\mu_1$

$\beta = P\{\overline{X} \text{ NOT falling in rejection region} \mid H_1\}$

$= P\{L \leq X \leq U \mid H_1: \overline{X} \sim N(\mu_1, (\sigma / \sqrt{n})^2)\}$

$= P\{\mu_0 - Z_{\alpha/2} \sigma / \sqrt{n} \leq \overline{X} \leq \mu_0 + Z_{\alpha/2} \sigma / \sqrt{n} \mid H_1\}$

$= Pr\left\{ \frac{(\mu_0 - Z_{\alpha/2} \sigma / \sqrt{n}) - \mu_1}{\sigma / \sqrt{n}} \leq \frac{\overline{X} - \mu_1}{\sigma / \sqrt{n}} \leq \frac{(\mu_0 + Z_{\alpha/2} \sigma / \sqrt{n}) - \mu_1}{\sigma / \sqrt{n}} \right\}$

$= \Phi\left( Z_{\alpha/2} - \frac{\delta \sqrt{n}}{\sigma} \right) - \Phi\left( -Z_{\alpha/2} - \frac{\delta \sqrt{n}}{\sigma} \right)$

Reject $H_0$ if

$\overline{X} < L = \mu_0 - Z_{\alpha/2} \sigma / \sqrt{n}$

$\overline{X} > U = \mu_0 + Z_{\alpha/2} \sigma / \sqrt{n}$

Note:

- The rejection region is designed using $H_0$ parameters!
- $\beta$ is calculated based on the test statistic following $H_1$ distribution!
Example

The mean contents of coffee cans filled on a particular production line are being studied. Standards specify that the mean contents must be 16.0 oz, and from past experience it is known that the standard deviation of the can contents is 0.1 oz. The hypotheses are

\[ H_0: \mu = 16.0 \]
\[ H_1: \mu \neq 16.0 \]

A random sample of nine cans is to be used, and type I error probability is specified as \( \alpha = 0.05 \). What is the type II error rate if the true mean contents are \( \mu_1 = 16.1 \text{ oz} \) or \( \beta(16.1) \)?

Given \( \sigma = 0.1, \mu_0 = 16, \mu_1 = 16.1, \delta = \mu_1 - \mu_0 = 0.1, n = 9, \alpha = 0.05 \)

\[ Z_{\alpha/2} = Z_{0.025} = \Phi^{-1}(0.975) = 1.96 \]

\[ \beta = \Phi\left( Z_{\alpha/2} - \frac{\delta \sqrt{n}}{\sigma} \right) - \Phi\left( -Z_{\alpha/2} - \frac{\delta \sqrt{n}}{\sigma} \right) = \Phi\left( 1.96 - \frac{0.1 \sqrt{9}}{0.1} \right) - \Phi\left( -1.96 - \frac{0.1 \sqrt{9}}{0.1} \right) \]

\[ \beta = \Phi(-1.04) - \Phi(-4.96) = 0.1492 = 14.92\% \]

Note: This formula can only be used to test the mean of a normal distribution with pre-known variance and two sided \( \alpha \) error rate (mean shift without a variance change).
Properties of type I & type II errors

• For a given sample size, one risk can only be reduced at the expense of increasing the other risk.

• For a given type I error rate, a desired type II error rate can be achieved by increasing the sample size at the price of an increased inspection cost.
ME 498 Manufacturing Data and Quality Systems

Methods and Philosophy of SPC
Basics of SPC

- **SPC is about controlling variations in products.**

- **Chance causes/common causes**
  - inherent variability (natural variation/background noise)
  
  Generally small and unavoidable!

- **Assignable causes/special causes**
  - problems arise in somewhat unpredictable fashion (operator errors, material defects, machine failures)

  Generally large. Corrective actions are needed.

![Graph showing normal distributions with chance and assignable causes](image-url)
Assignable causes

Assignable cause 1:
change setup

Assignable cause 2:
worn tools

Assignable cause 3:
worn tools + shift

Normal operation

\[ \mu_0, \sigma_0 \]

\[ \mu_0, \sigma_1 > \sigma_0 \]

\[ \mu_2 < \mu_0, \sigma_1 > \sigma_0 \]

\[ \mu_1 > \mu_0, \sigma_0 \]
Statistical in-control and out-of-control process

**In-control process:**
A process operating with only chance causes is said to be in-control

Chance causes $\iff$ In-control process $\iff$
Distribution is unchanged
(Both mean & variance are constant)

**Out-of-control process:**
A process operating in the presence of assignable causes is said to be out-of-control

Assignable causes $\iff$ Out-of-control process $\iff$
Distribution is changed
(Either mean or variance, or both are changed)
Objectives of SPC

1. Monitoring the process and detecting process changes  ➔ Control charts

2. Diagnosing the assignable causes  ➔
   - Analysis of process change patterns
   - Cause-effect analysis
   - Remove root causes
   - SPC + APC
   - Robust design

3. Providing corrective action plans  ➔

```
<table>
<thead>
<tr>
<th>Process Variables</th>
<th>Output Product Quality Characteristic</th>
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```

```
| Take Action       | 3. Formulate Actions                  |
|                   | 2. Find Root Causes                   |
|                   | 1. SPC Monitoring                     |
```

- "statistically out of control"
- Assignable Causes
- Chance Causes
  - "statistically in control"
Implementing SPC: magnificent SEVEN tools

1. Histogram
2. Check Sheet
3. Pareto Chart
4. Cause and Effect Diagram (Fishbone diagram)
5. Scatter Diagram
6. Defect Concentration Diagram
7. Control Chart
Check sheet

- **A simple tool for data collection**
- **Give a time-oriented summary of historical data**
- **Looking for trend and/or other meaningful patterns**

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Vilfredo Pareto (Economist) observed in 1906 that 80% of the land in Italy was owned by 20% of the population.

Pareto principle (a.k.a. 80-20 rule, law of vital few): 80% of the problems are caused by 20% of the causes.

Another way to visualize data.
Cause-and-effect diagram for the tank defect problem
Cause-and-effect diagram

How to Construct a Cause-and-Effect Diagram

1. Define the problem or effect to be analyzed.

2. Form the team to perform the analysis. Often the team will uncover potential causes through brainstorming.

3. Draw the effect box and the center line.

4. Specify the major potential cause categories and join them as boxes connected to the center line.

5. Identify the possible causes and classify them into the categories in step 4. Create new categories, if necessary.

6. Rank order the causes to identify those that seem most likely to impact the problem.

7. Take corrective action.
Scatter diagram

Showing inter-relationships between two different variables

Correlation does NOT imply the causal relationship between two variables

\[
\text{Correlation coefficient:}
\hat{\rho} = \frac{\sum_{i=1}^{n} (x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum_{i=1}^{n} (x_i - \bar{x})^2 \times \sum_{i=1}^{n} (y_i - \bar{y})^2}}
\]
Defect concentration diagram

Identify particular locations and link with potential causes.

- Surface-finish defects on a refrigerator.
- Showing various types of defects drawn on different views.
- Engine surface measurement.
- Showing the spatial distribution of out-of-limit points.
Defect concentration diagram

- Rental car checklist
- Visual inspection of surface damage

- Check sheet in shirt manufacturing
- Locations of defects are marked
Control charts: objectives

- Monitoring the process and detecting process changes.
- Estimating the process mean and variance.
- Guiding the decisions on adjusting the process mean and reducing process variation.
Control charts: concepts

\[ \mu = \mu_0 \] \hspace{1cm} \text{In-control process}

\[ \mu = \mu_1 \] \hspace{1cm} \text{Out-of-control process}

Have we seen this before somewhere?
Control charts: concepts

- **Control Chart**: is a graphical display of a quality characteristic that has been measured or computed from a sample versus the sample number or time.
  - **Center Line (CL)** – represents the average value of the quality characteristic corresponding to the in-control state (only chance causes are present).
  - **Upper Control Limit (UCL), Lower Control Limit (LCL)** – are chosen so that if the process is in control, nearly all of the sample points will fall between them.
Out-of-control action plan (OCAP)

- **Out-of-control action plan**: an OCAP is a flowchart or text-based description of the sequence of activities that must take place following the occurrence of an out-of-control signal.

Process improvement using control charts
Announcements

• Homework 1 graded and returned. Solutions posted.
• Homework 2 assigned.
• Homework guidelines:
  ➢ A complete submission features the following items: (a) a brief written/typeset report including all figures and results, and explanations of necessary steps taken to obtain them; and (b) and the source code (Python is recommended).
  ➢ Item (a) shall be submitted in hard copies, and Item (b) shall be submitted in a zipped folder, which is named as your_net_id_here.zip, through Compass.
• Software: you are strongly suggested to use Python for homework.
Type II Error: test mean of a normal population with known $\sigma$

$$H_0 : \mu = \mu_0$$

$$H_1 : \mu \neq \mu_0$$

$$H_1 : \mu = \mu_1 = \mu_0 + \delta; (\delta \neq 0)$$

$$\beta = P\{\text{test statistic NOT falling in rejection region} \mid H_0 \text{ is NOT true}\}$$

$\beta$ can only be calculated for a given $\mu_1$

$$\beta = \Pr\{\bar{X} \text{ NOT falling in rejection region} \mid H_1\}$$

$$= \Pr\{L \leq \bar{X} \leq U \mid H_1 : \bar{X} \sim N(\mu_1, (\sigma / \sqrt{n})^2)\}$$

$$= \Pr\{\mu_0 - Z_{\alpha/2} \sigma / \sqrt{n} \leq \bar{X} \leq \mu_0 + Z_{\alpha/2} \sigma / \sqrt{n} \mid H_1\}$$

$$= \Pr\left\{\frac{(\mu_0 - Z_{\alpha/2} \sigma / \sqrt{n}) - \mu_1}{\sigma / \sqrt{n}} \leq \frac{\bar{X} - \mu_1}{\sigma / \sqrt{n}} \leq \frac{(\mu_0 + Z_{\alpha/2} \sigma / \sqrt{n}) - \mu_1}{\sigma / \sqrt{n}}\right\}$$

$$= \Phi \left( Z_{\alpha/2} - \frac{\delta \sqrt{n}}{\sigma} \right) - \Phi \left( -Z_{\alpha/2} - \frac{\delta \sqrt{n}}{\sigma} \right)$$

Reject $H_0$ if

$$\bar{X} < L = \mu_0 - Z_{\alpha/2} \sigma / \sqrt{n}$$

$$\bar{X} > U = \mu_0 + Z_{\alpha/2} \sigma / \sqrt{n}$$

Note:

- The rejection region is designed using $H_0$ parameters!
- $\beta$ is calculated based on the test statistic following $H_1$ distribution!
\( \beta \) error rate

\[
\beta = \Phi \left( Z_{\frac{\alpha}{2}} - \frac{\delta \sqrt{n}}{\sigma} \right) - \Phi \left( -Z_{\frac{\alpha}{2}} - \frac{\delta \sqrt{n}}{\sigma} \right)
\]
Construction of control charts

Let $W$ denote the monitoring statistic of a quality characteristic, and assume $W \sim NID(\mu_w, \sigma_w^2)$ when the process is in-control.

Shewhart Control Chart:

\[
\begin{align*}
UCL &= \mu_w + k\sigma_w \\
CL &= \mu_w \\
LCL &= \mu_w - k\sigma_w
\end{align*}
\]

(K-sigma limits)

- $k$ is the distance of control limits from the center line, expressed in standard deviation units.

- $k = 3$ is usually used for control limits
  - approximately 99.73% of the in-control data lies within the 3-sigma limits ($\alpha = 0.0027$)
- $k = 2$ is usually used for warning limits
  - increase the sensitivity of the control chart
  - result in an increased risk of false alarms

Pr($\mu-\sigma \leq x \leq \mu+\sigma$) = 68.26%
Pr($\mu-2\sigma \leq x \leq \mu+2\sigma$) = 95.46%
Pr($\mu-3\sigma \leq x \leq \mu+3\sigma$) = 99.73%
Example: piston rings

The inner-diameter of piston rings follows a normal distribution with mean = 75 mm, and variance = 9.

a) What is the sampling distribution of sample average (denoted as \( \bar{X} \)) with sample size \( n=5 \)?

b) Construct a monitoring chart on the sample average (called X-bar control chart) with sample size \( n=5 \) and \( k=3 \).
Example: piston rings

V - The cylinders are arranged in two banks set at an angle to one another.
Example: piston rings

The inner-diameter of piston rings follows a normal distribution with mean = 75 mm, and variance = 9.

a) What is the sampling distribution of sample average (denoted as X-bar) with sample size n=5?

b) Construct a monitoring chart on the sample average (called X-bar control chart) with sample size n=5 and k=3.

\[ a) \text{ The } j\text{th observation of sample } i: X_i \sim N\left(\mu_0, \sigma_0^2\right) \]

Sample i's average: \[ \bar{X}_i = \frac{\sum_{j=1}^{n} X_{ij}}{n} \sim N\left(\mu_0, \frac{\sigma_0^2}{n}\right) \]

\[ b) \mu_0 = 75, \sigma_0 = \sqrt{9} = 3, \ n = 5, \ k = 3 \]

\[ \bar{X} \text{ control chart: } \left\{\begin{array}{l}
UCL_{\bar{X}} = \mu_{\bar{X}} \pm k\sigma_{\bar{X}} = \mu_0 \pm k \frac{\sigma_0}{\sqrt{n}} = 75 \pm 3 \times \frac{3}{\sqrt{5}} = 79.02 \\
LCL_{\bar{X}} = \mu_{\bar{X}} = 75 \\
CL_{\bar{X}} = \mu_{\bar{X}} = 75
\end{array}\right. \]
Control charts vs. hypothesis testing

**Hypothesis Testing**

H$_0$ is rejected if the test statistic lies in the rejection region

\[ \begin{cases} 
H_0 : \mu = \mu_0 & \text{In-control process} \\
H_1 : \mu \neq \mu_0 & \text{Out-of-control process} 
\end{cases} \]

\[ |Z_0| > Z_{\alpha/2} \]

**Control Charts**

- Control chart has UCL & LCL, and shows the time series of the data.
- The process is out-of-control if the data is plotted beyond the control limits.

\[
\bar{x} \sim N(\mu, \sigma^2/n) \\
UCL = \mu + k \frac{\sigma}{\sqrt{n}} \\
CL = \mu \\
LCL = \mu - k \frac{\sigma}{\sqrt{n}} \\
\]

**Main difference**

In control charts, we also consider a non-random pattern of the plotted points as out-of-control even though all points fall within the control limits.

*Time order matters!*
Why time order matters in control charts

(a) Non-random pattern, with assignable causes

(b) Non-random pattern disappears

**Conclusion:** Control chart showing the time sequence of sample data can reveal a non-random pattern for early detection of assignable causes, while the hypothesis testing of non-ordered samples cannot do this.
Type I and type II errors in hypothesis testing

**Decision based on samples**

<table>
<thead>
<tr>
<th>Reality</th>
<th>Reject $H_0$</th>
<th>NOT Reject $H_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$H_0$ is True $\mu = \mu_0$</td>
<td>$</td>
<td>Z_0</td>
</tr>
<tr>
<td>$H_0$ is NOT True $\mu \neq \mu_0$</td>
<td>Power $(1 - \beta)$</td>
<td>Type II error $(\beta)$</td>
</tr>
</tbody>
</table>

**Reality**
- $H_0$ is True $\mu = \mu_0$
- $H_0$ is NOT True $\mu \neq \mu_0$

**Type I error** $(\alpha)$

**Type II error** $(\beta)$

**Confidence** $(1 - \alpha)$

**Power** $(1 - \beta)$

**Decision based on samples**

- **Rejection region**
  - $H_0$ rejection region: $L$ to $U$
  - $H_1$ rejection region: $\mu_1$ to $x$

- **Alpha** $(\alpha)$
  - $\alpha/2$

- **Beta** $(\beta)$
  - $\beta$
Type I and type II errors in control charts

### Decision based on monitoring statistic

<table>
<thead>
<tr>
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<th>Control chart states that the process is out-of-control</th>
<th>Control chart states that the process is in-control</th>
</tr>
</thead>
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<td>Process is in-control</td>
<td>Type I error ((\alpha))</td>
<td>Confidence ((1 - \alpha))</td>
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![Diagram showing control charts and decision making](image)
Type I error of control chart

\[ \alpha = \Pr\{\text{type I error}\} = \Pr\{\text{Control chart states the process is out-of-control | Process is actually in-control}\} = \Pr\{A \text{ point is plotted beyond the control limits | Process is actually in-control}\} \]

Example:
Type I error of a X-bar chart with known \( \sigma \)

\[ X \sim N(\mu_0, \sigma_0^2) \rightarrow \bar{X} \sim N\left(\mu_0, \frac{\sigma_0^2}{n}\right) \rightarrow \begin{cases} UCL \bar{X} = \mu_0 + k \frac{\sigma_0}{\sqrt{n}} \\ LCL \bar{X} = \mu_0 - k \frac{\sigma_0}{\sqrt{n}} \end{cases} \]

\[ \alpha = \Pr\{\bar{X} < LCL \text{ or } \bar{X} > UCL \} = 1 - \Pr\{LCL \leq \bar{X} \leq UCL \} = 2\left[1 - \Phi(k)\right] \]

Note: For \( k \)-sigma control charts, type I error only depends on \( k \).
Type II error of control chart

\[ \beta = \text{Pr\{type II error\}} \]

\[ = \text{Pr\{Control chart states the process is in-control | Process is actually out-of-control\}} \]

\[ = \text{Pr\{A point is plotted within the control limits | Process is actually out-of-control\}} \]

\[ \mu = \mu_0 \quad \text{In-control} \]

\[ \mu = \mu_1 = \mu_0 + \delta; (\delta \neq 0) \quad \text{Out-of-control} \]

Example: Type II error in X-bar chart with known \( \sigma \)

\[ H_0 : X \sim N(\mu_0, \sigma_0^2) \]

\[ H_1 : X \sim N(\mu_1, \sigma_0^2) \]

\[ \delta = \mu_1 - \mu_0 \]

K-sigma control limits

\[ \begin{align*}
UCL &= \mu_0 + k \sigma_0 / \sqrt{n} \\
LCL &= \mu_0 - k \sigma_0 / \sqrt{n}
\end{align*} \]

Type II error depends on \( k, \delta, \text{ and } n \).
Example: piston rings

The inner-diameter of piston rings follows a normal distribution with mean = 75 mm, and variance = 9.

a) Construct a X-bar Shewhart control chart with n = 16 and k = 3.

b) If the mean of process changes to 78 mm, calculate the Type II error.

c) What is the probability of detecting this mean shift before plotting the 4th sample?

\[ a) \quad \mu_0 = 75, \quad \sigma_0 = \sqrt{9} = 3, \quad n = 16, \quad k = 3; \]

\[ b) \quad \mu_1 = 78 \]

\[ \beta = \Pr\{72.75 \leq \bar{X} \leq 77.25 \mid \bar{X} \sim N(78,3^2/16)\} \]

\[ c) \quad \Pr\{\text{detect the change before the 4th sample}\} = 1 - \Pr\{\text{all three samples do not detect the change}\} \]
Probability limits

How to calculate $p$-probability limits ($0 < p < 0.5$):

$$UPL_w \Rightarrow UCL_w = \mu_w + k\sigma_w = \mu_w \pm Z_{\alpha/2}\sigma_w = \mu_w \pm Z_p\sigma_w$$

$$LPL_w \Rightarrow LCL_w$$

$H_0 : W \sim NID(\mu_w, \sigma_w^2)$

- $\Pr(W > UPL) = p$  \textbf{UPL: Upper Probability Limit}
- $\Pr(W < LPL) = p$  \textbf{LPL: Lower Probability Limit}

Probability limits offer more flexibility than $k\sigma$ control limits.
Example: piston rings

• The inner-diameter of piston rings follows a normal distribution with mean = 75 mm, and variance = 9. Calculate the 0.001 probability limits for individual observations (n = 1).

\[ \mu_0 = 75, \quad \sigma^2 = 9, \quad n = 1 \]

\[ W = X \sim N(75, \ 3^2) \]

\[
\begin{align*}
UPL_W &= \mu_w \pm z_p \sigma_w = 75 \pm z_{0.001} \times 3 = 75 \pm 3.09 \times 3 = 84.27 \\
LPL_W &= \mu_w \pm z_p \sigma_w = 75 \pm z_{0.001} \times 3 = 75 \pm 3.09 \times 3 = 65.73 \\
\end{align*}
\]

\[ z_{0.001} = \Phi(1 - 0.001) = 3.09 \]
Performance assessment of control charts

- **Based on the probability in OC curves**
  - Fix type I error rate and compare type II error rate (or detection power)

- **Based on average run length (ARL)**
Average run length

- Run length (RL): The number of samples/points that are plotted until a point is out of control limits.

- RL is a random variable.

- What is the distribution of RL?

- What is the mean of RL?
Average run length

- The number of samples/points that are plotted until a point is out of control limits, follows a Geometric Distribution.

- **ARL:** The average number of samples/points that are plotted until a point is out of control limits.

\[
ARL = \frac{1}{p} = \begin{cases} 
ARL_0 = \frac{1}{\alpha} & \text{If the process is actually in-control} \\
ARL_1 = \frac{1}{1 - \beta} & \text{If the process is actually out-of-control}
\end{cases}
\]

- Example: for 3-sigma control limits \((\alpha=0.0027)\)
  \[
  ARL_{\text{in-control}} = ARL_0 = ARL_{\alpha} = ARL_0 = \frac{1}{\alpha} = \frac{1}{0.0027} = 370.
  \]

**Comments:** Even the process is in-control, an out-of-control sample will be generated every 370 samples on average.
Example

• Suppose that a control chart with 2-sigma limits is used to control a process. If the process remains in control, find the average run length until a false out-of-control signal is observed. Compare this with the in-control ARL for 3-sigma limits.

• Solution: For K=2 sigma control limits,

\[ \alpha = 2[1 - \Phi(k)] = 2[1 - \Phi(2)] = 0.0455 \]

\[ \Phi(2) = 0.97725 \]

\[ ARL_0 = \frac{1}{\alpha} = 22 \]
Example

Suppose that a X-bar control chart with 3-sigma limits and n=4 is used to control a process. Assume the process mean is changed by 1 standard deviation.

(a) What is the probability of detecting the change by the first sample after shift?

(b) What is the probability of detecting the change before collecting the 5th sample?

(c) What is the average number of points plotted on the chart until a signal is observed?

\[ \beta = \Phi \left( k - \frac{\delta}{\sigma_0 / \sqrt{n}} \right) - \Phi \left( -k - \frac{\delta}{\sigma_0 / \sqrt{n}} \right) = \Phi \left( 3 - \frac{1\sigma_0}{\sigma_0 / \sqrt{4}} \right) - \Phi \left( -3 - \frac{1\sigma_0}{\sigma_0 / \sqrt{4}} \right) \]

\[ = \Phi(1) - \Phi(-5) = 0.8413 - 0 = 0.8413 \]

\[ 1 - \beta = 1 - 0.8413 = 0.1587 \]

(b) \( \text{Prob} = 1 - \text{Pr(fail to detect based on all 4 samples)} = 1 - \beta^4 = 0.4990 \)

(c) \( \text{ARL}_{1-\beta} = \text{ARL}_1 = \text{ARL}_{\text{out of control}} = \frac{1}{1 - \beta} = 6.30 \)
Rules to signal: detect out-of-control status

Rules to signal: an out-of-control status

• Rule 1: When one or more points fall beyond the control limits

• Rule 2: Sample points exhibit any nonrandom pattern of behaviors
  (additional capability of control charts beyond hypothesis tests)

Description of nonrandom patterns

• The location of the current point can be approximately predicted by previous points

  ➢ Run up: a sequence of increasing observations
  ➢ Run down: a sequence of decreasing observations
  ➢ Cyclic pattern
  ➢ etc.
Examples of nonrandom patterns

• Pattern is very nonrandom
• 19 of 25 points plot below the center line, while only 6 plot above
• Following 4\textsuperscript{th} point, 5 points in a row increase in magnitude \(\rightarrow\) “run up”
• There is also an unusually long \textit{run down} beginning with 18\textsuperscript{th} point
Nonrandom patterns and common causes

• **Jumps in process level**
  1. New supplier
  2. New worker
  3. New machine
  4. New technology
  5. Change in method or process
  6. Change in inspection device or method

• **High proportion of points near outer limits**
  1. Over control
  2. Large difference in material quality and test method
  3. Control of 2 or more processes on one chart
  4. Mixtures of materials of different quality
  5. Multiple charters
  6. Improper subgrouping
Nonrandom patterns and common causes

• **Recurring cycles**
  1. Temperature and other cyclic environmental effects
  2. Worker fatigue
  3. Differences in measuring devices used in order
  4. Regular rotation of machines or operators
  5. Scheduled preventive maintenance (R chart)
  6. Tool wear (R chart)

• **Trends**
  1. Gradual equipment deterioration
  2. Worker fatigue
  3. Accumulation of waste products
  4. Improvement or deterioration of worker skill/effort (especially in R chart)
  5. Drift in incoming materials quality

• **Stratification** (lack of variability)
  1. Incorrect calculation of control limits
  2. Systematic sampling
Western electric rules (zone rules charts)

1. One point plots outside the three-sigma control limits,
2. Two out of three consecutive points plot beyond the two-sigma warning limits,
3. Four out of five consecutive points plot at a distance of one-sigma or beyond from the center line, or
4. Eight consecutive points plot on one side of the center line.
More sensitizing rules

Some Sensitizing Rules for Shewhart Control Charts

Standard Action Signal:

1. One or more points outside of the control limits.
2. Two of three consecutive points outside the two-sigma warning limits but still inside the control limits.
3. Four of five consecutive points beyond the one-sigma limits.
4. A run of eight consecutive points on one side of the center line.
5. Six points in a row steadily increasing or decreasing.
6. Fifteen points in a row in zone C (both above and below the center line).
7. Fourteen points in a row alternating up and down.
8. Eight points in a row on both sides of the center line with none in zone C.
9. An unusual or nonrandom pattern in the data.
10. One or more points near a warning or control limit.
Combining type I error rate of multiple sensitizing rules

- Decision to signal: the process is concluded out of control if any one of the rules is applied.

If all $k$ rules are independent,

$$\alpha = 1 - \prod_{i=1}^{k} (1 - \alpha_i)$$

$\alpha_i$ is type I error rate of using Rule $i$ alone.
Control limits vs. specification limits

1. Control Limits are used to determine if a process is in-control
   (if any distribution parameters of process/product measurements’ are changed, e.g. mean or variance).

   **Statistical tests:** determined based on collected samples from a process

2. Specification Limits are used to determine if a product will function in the intended design fashion.

   **Design requirement:** determined by customer needs and standards

   - **LCL**
   - **UCL**
   - **LSL**
   - **USL**

   Monitoring statistic W (e.g. X-bar or R) for detecting process distribution change (mean or variance change for the normal distribution)

   ![Diagram](image)
Examples: true or false

• One of the main purpose of a control chart is to distinguish chance and assignable causes. [True]

• If we use k=2 instead of k=3 for k-sigma control limits, the type I error will decreases. [False]

• If the process consistently produces products outside of the engineering specification limits, the process is out of control. [False]

• When designing a process control chart, by increasing its type I error, we can always increase its detection power for a given mean shift. [True]

• A 3-sigma limits X-bar chart will always have a type I error rate equal to 0.27%. [True]
A x-bar control chart with 3-sigma control limits is used to monitor process mean. Inspection decision is made based on two successive samples using the following rules: (n = 4)

Rule 1: If one or two sample means exceed either the upper or lower control limit

Rule 2: If two sample means fall on the same side of the center line

(1) What is type I error rate using Rule 1 alone?

(2) What is type I error rate using Rule 2 alone?

(3) What is the overall type I error rate based on the two rules given that the two rules are independent?

(4) If the process has a mean shift of one process standard deviation, what is type II error rate using Rule 1?

(5) If the process has a mean shift of one process standard deviation, what is type II error rate using Rule 2?
(1) $\alpha_{rule1} = \Pr(\text{any sample's mean exceeds control limits } | \ H_0 )$
\[ = 1 - \Pr(\text{both sample means fall within LCL and UCL } | \ H_0 ) \]
\[ = 1 - \left[ \Pr(\text{a sample mean falls within LCL and UCL } | \ H_0 ) \right]^2 \]
\[ = 1 - \left[ \Pr(\ \text{LCL} \leq \bar{X} \leq \text{UCL} \ | \ H_0 ) \right]^2 \]
\[ = 1 - \left( 1 - \alpha_{1sample} \right)^2 \]
\[ = 1 - (1 - 0.0027)^2 \]
\[ \alpha_{1sample} = 0.0027 \text{ in 3-sigma limits control chart} \]
\[ = 0.0054 \]

(2) $\alpha_{rule2} = \Pr(\text{2 sample means fall on the same side of the center line } | \ H_0 )$
\[ = \Pr(\text{2 sample means above CL } | \ H_0 ) + \Pr(\text{2 sample means below CL } | \ H_0 ) \]
\[ = \left[ \Pr(\bar{X} > CL \ | \ H_0 ) \right]^2 + \left[ \Pr(\bar{X} < CL \ | \ H_0 ) \right]^2 \]
\[ = 0.5^2 + 0.5^2 = 0.5 \]

(3) $\alpha_{overall} = 1 - (1 - \alpha_{rule1})(1 - \alpha_{rule2}) = 1 - (1 - 0.0054)(1 - 0.5) = 0.5027$
(4) \( \beta_{rule1} = \Pr(\text{both sample means fall within LCL and UCL} \mid H_1) \)

\[
= \left[ \Pr(LCL \leq \bar{X} \leq UCL \mid H_1) \right]^2
= \beta_{sample}^2
\]

\[\beta_{sample} = \Phi(k - \frac{\delta}{\sigma / \sqrt{n}}) - \Phi(-k - \frac{\delta}{\sigma / \sqrt{n}}) = \Phi(3 - \frac{1\sigma}{\sigma / \sqrt{4}}) - \Phi(-3 - \frac{1\sigma}{\sigma / \sqrt{4}})\]

\[= \Phi(1) - \Phi(-5) = 0.8413\]

\[\beta_{rule1} = \beta_{sample}^2 = 0.8413^2 = 0.7079\]

(5) \( \beta_{rule2} = 1 - \Pr(2 \text{ sample means fall on the same side of the center line} \mid H_1) \)

\[= 1 - \Pr(2 \text{ sample means above CL} \mid H_1) - \Pr(2 \text{ sample means below CL} \mid H_1)\]

\[= 1 - \left[ \Pr(\bar{X} > CL \mid H_1) \right]^2 - \left[ \Pr(\bar{X} < CL \mid H_1) \right]^2\]

\[P_L = \Pr(\bar{X} < CL \mid H_1)\]

\[= P(\bar{X} < \mu_0 \mid \mu_1 = \mu_0 + \sigma) = P\left(\frac{\bar{X} - \mu_1}{\sigma / \sqrt{n}} < \frac{\mu_0 - \mu_1}{\sigma / \sqrt{n}}\right) = \Phi\left(\frac{\mu_0 - \mu_0 - \sigma}{\sigma / \sqrt{4}}\right) = \Phi(-2) = 0.02275\]

\[P_U = \Pr(\bar{X} > CL \mid H_1)\]

\[= P(\bar{X} > \mu_0 \mid \mu_1 = \mu_0 + \sigma) = 1 - P(\bar{X} \leq \mu_0 \mid \mu_1 = \mu_0 + \sigma) = 1 - \Phi(-2) = 0.97725\]

\[\beta_{rule2} = 1 - P_L^2 - P_U^2 = 0.0445\]