Entanglement, discord and other quantum correlations

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Entanglement, use in quantum information, decay and restoration;
Bell Inequalities; Entropic Inequalities;
Quantum Discord, entropic and geometric;
Calculations for 7-parameter X-states of two qubits;
X- and extended X-states, qubit-qudit or N qubits;
Geometrical generalization of Bloch Sphere to two-qubit states and their evolution : a decomposition of SU(N) analogous to that of SU(2), provides a minimal unitary operator useful in calculating discord

Dmitry Uskov, Sai Vinjanampathy, Mazhar Ali & Gernot Alber

ARPR arXiv:1701.05922

Entanglement and other quantum correlations

Quantum Information:

Computing, Cryptography, Teleportation

Basic Quantum Principles:

Superposition -- linearity of QM

Entanglement -- non-separability of subsystems

other correlations: discord, inequalities,..

Identity of particles -- spin/statistics

Applications in Quantum Information

Quantum Correlations

Entanglement: Separability Mixed State ρ

Concurrence Positivity of Partial Transpose

qubit-qubit & qubit-qutrit, qudit-quDit ?

Quantum Discord, based on Entropy

Mutual Information – Max Classical Correlation

- when one side is a qubit
- X states qubit-qubit

  symmetry structure
- Unitary evolution for general SU(N)
- Discord for general dimension, X states

Ollivier and Zurek, Phys. Rev. Lett. **88**, 017001 (2002)
Henderson and Vedral, J. Phys. A **34**, 6899 (2002)

Quantum Discord   Entropy $S(\rho) = -\mathrm{tr}\,(\rho\,\log_2 \rho)$

Pure states

$$|\uparrow\uparrow\rangle \quad \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \rho^{A,B} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$(|\uparrow\rangle + |\downarrow\rangle)^A (|\uparrow\rangle + |\downarrow\rangle)^B / 2$$

$$\frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \quad \frac{1}{4}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \quad \rho^{A,B} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

All eigenvalues 1 and 0, all entropies 0   unentangled states

Pure entangled Bell state $(|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle)/\sqrt{2}$

$$|AB\rangle \qquad\qquad \rho^{AB}$$

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & \pm 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \pm 1 & 0 & 0 & 1 \end{pmatrix} \quad \rho^{A,B} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

mixed

Quantum Mutual Information $\quad I = S^A + S^B - S^{AB}$

$$S^A = 2(-\frac{1}{2}\log_2 \frac{1}{2}) = 1 \qquad I = 2$$

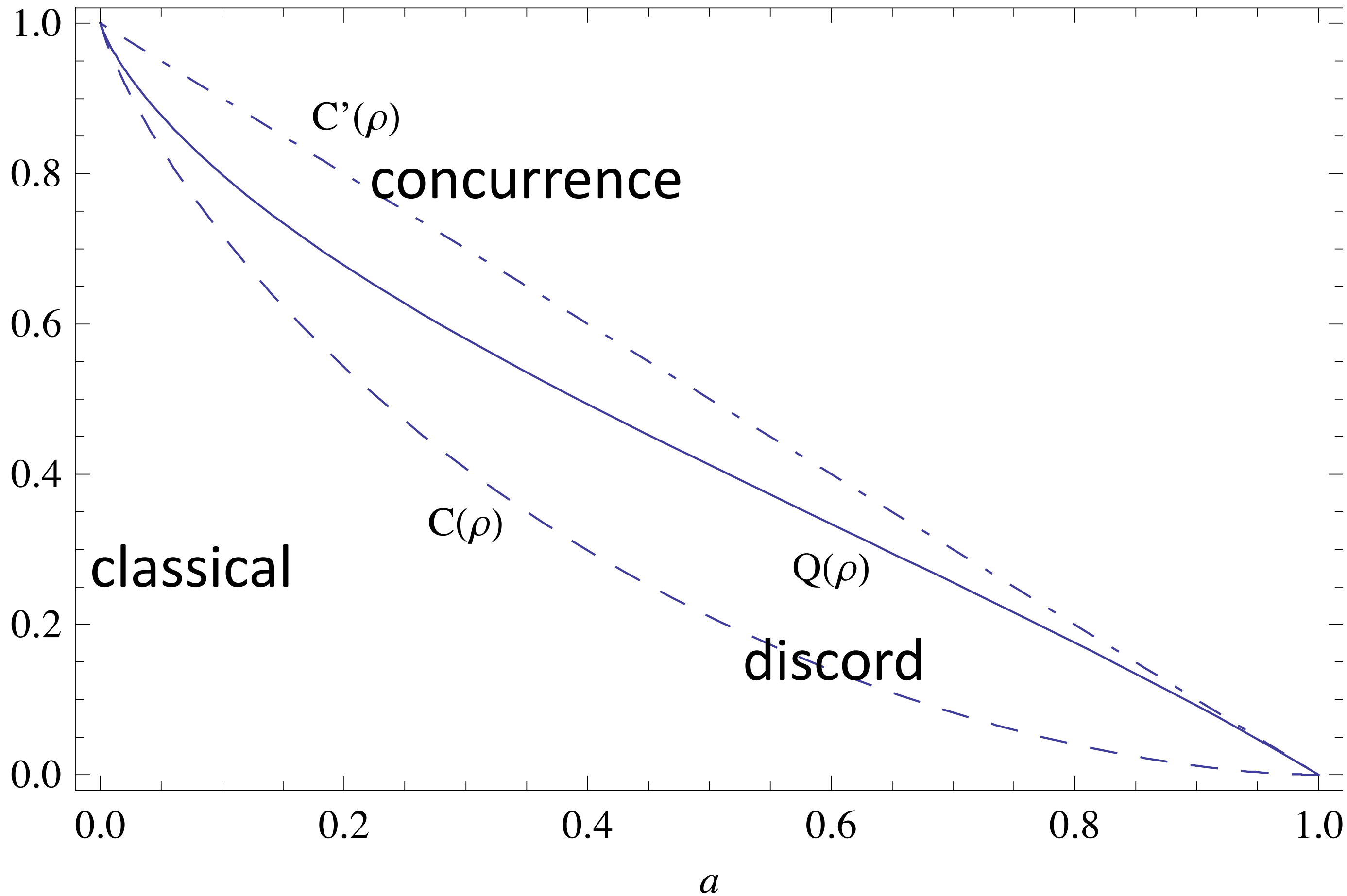
Classical correlation C: max information on A by measurements on B $\quad$ C = 1

Quantum Discord $\quad Q = I - C$

$$\rho = (1-a)|S^+\rangle\langle S^+| + a|\downarrow\downarrow\rangle\langle\downarrow\downarrow|$$

Mazhar Ali

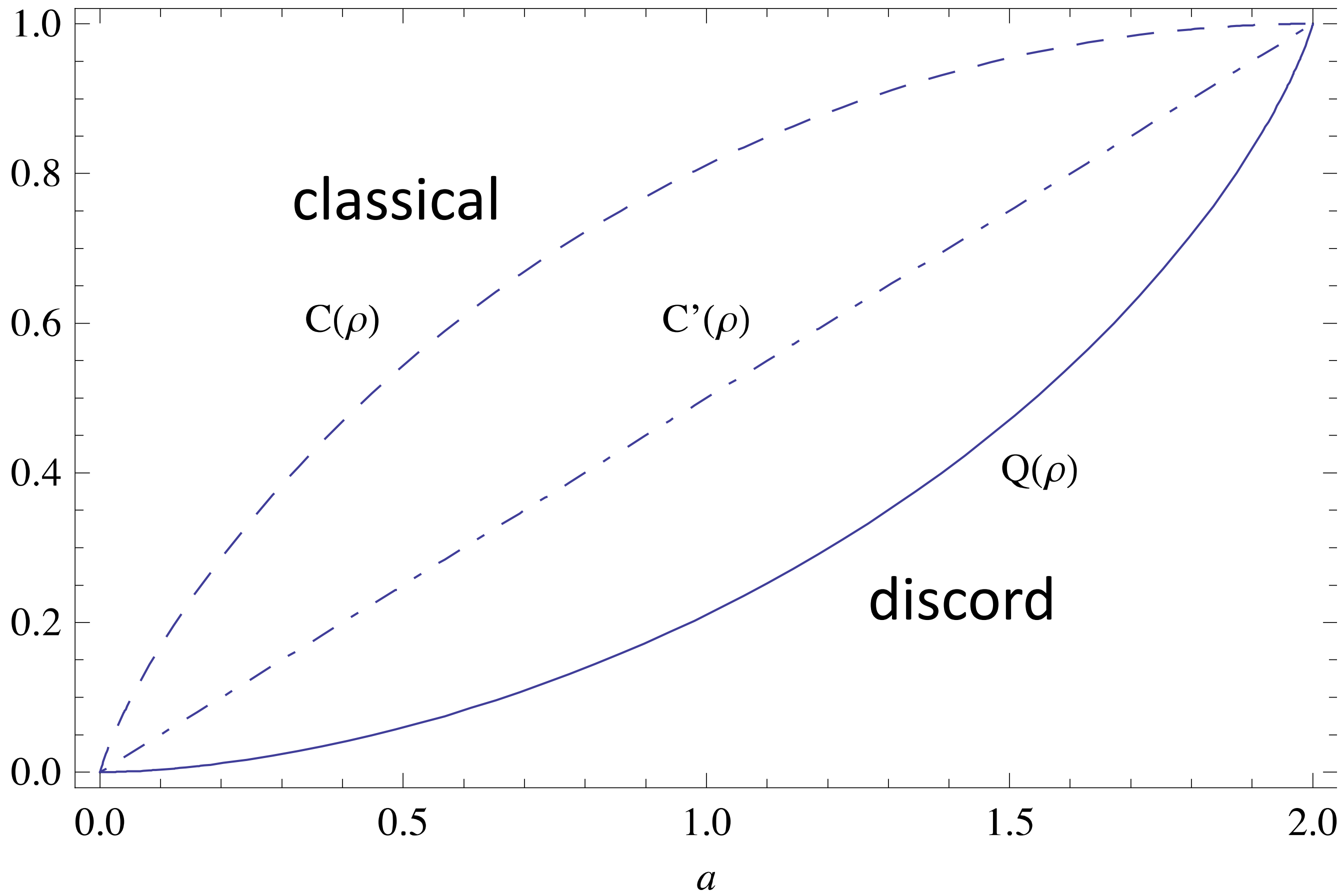
$$|S^+\rangle = |\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle$$



M. Ali, ARPR, G. Alber, Phys. Rev. A **81**, 042105 (2010)

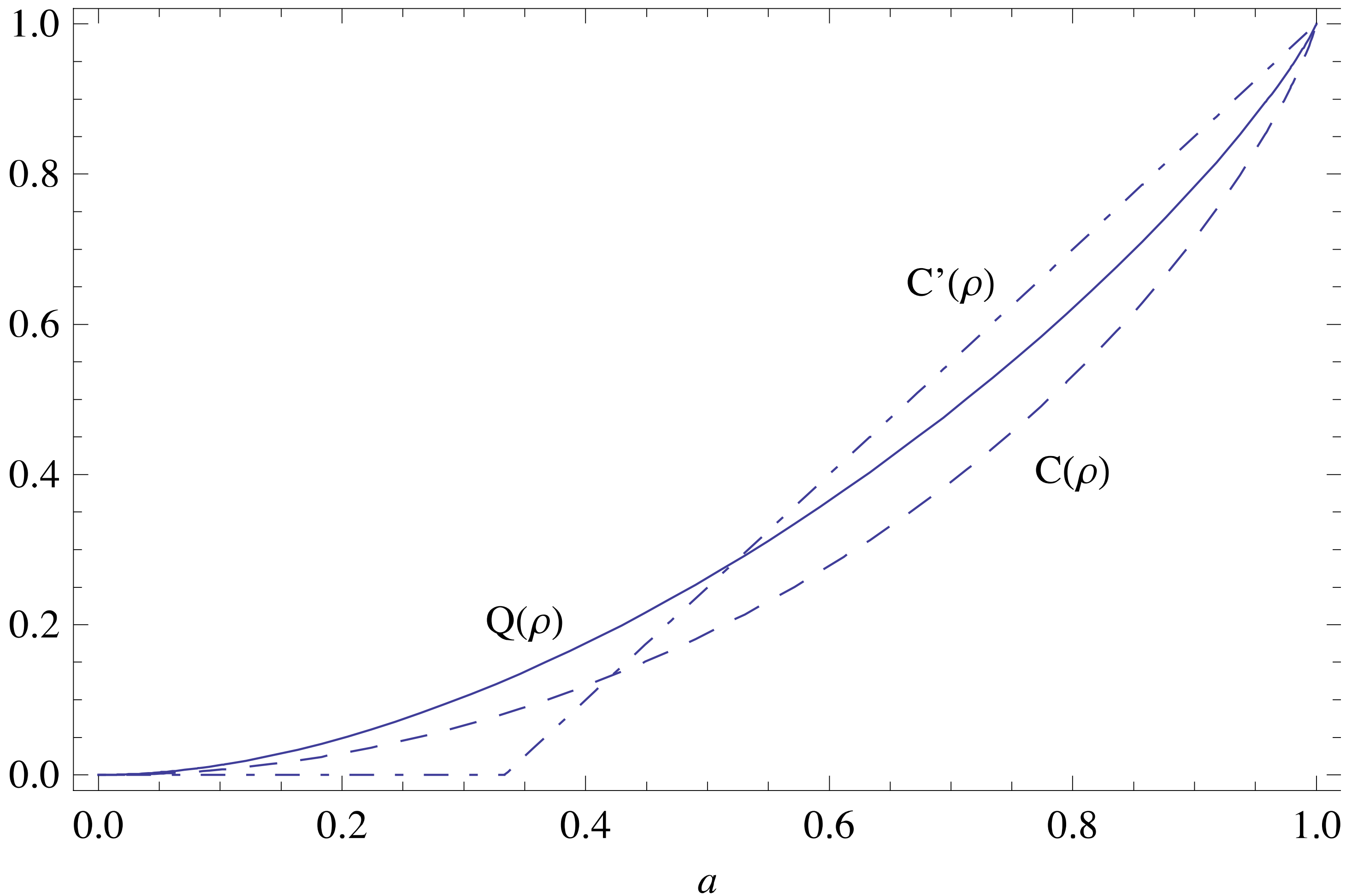
$$\rho = a|P^+\rangle\langle P^+| + (1-a)|\downarrow\downarrow\rangle\langle\downarrow\downarrow|$$

$$|P^+\rangle = |\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle$$



Werner state   separable for a < 1/3

$$\rho = a|S^-\rangle\langle S^-| + \mathcal{I}(1-a)/4$$



Quantum mutual information

$$\mathcal{I}(\rho^{AB}) = S(\rho^{A}) + S(\rho^{B}) - S(\rho^{AB})$$

$$S(\rho) = -\mathrm{tr}\,(\rho\,\log_2 \rho)$$

$$\rho_i = \frac{1}{p_i}(A_i \otimes I)\rho(A_i \otimes I)$$

p_i probabilities, trace

S. Luo, Phys. Rev. A **77**, 042303 (2008)

Conditional density operator upon measurements over A

$$A_i = U\,\Pi_i\,U^\dagger \qquad U = t\,I + \mathrm{i}\,\vec{y}\cdot\vec{\sigma}$$

$$\Pi_i = |i\rangle\langle i|,\ i = \pm$$ von Neumann projectors

Conditional density matrix for B because of measurements over A:

$$S(\rho|\{A_i\}) := \sum_i p_i\,S(\rho_i)$$

Its quantum mutual information: $\mathcal{I}(\rho|\{A_i\}) := S(\rho^B) - S(\rho|\{A_i\})$

$$\mathcal{C}(\rho) := \sup_{\{A_i\}} \mathcal{I}(\rho|\{A_i\}) \qquad \mathcal{Q}(\rho) := \mathcal{I}(\rho) - \mathcal{C}(\rho)$$ Discord

For A a qubit: $A_\pm = (I \pm \vec{\sigma}\cdot\vec{z})/2$

$$A_+ = \begin{pmatrix} \cos^2(\theta/2) & \frac{1}{2}\sin\theta\exp(-i\phi) \\ \frac{1}{2}\sin\theta\exp(i\phi) & \sin^2(\theta/2) \end{pmatrix}$$

$$\vec{z} = \{2(-ty_2 + y_1y_3), 2(ty_1 + y_2y_3), t^2 + y_3^2 - y_1^2 - y_2^2\}$$

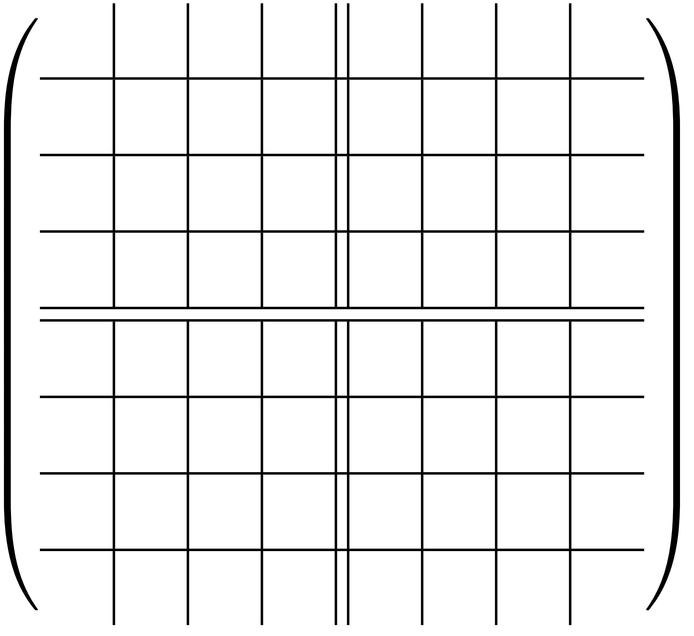
A_- parity conjugate $(\theta, \phi) \rightarrow (\pi - \theta, \pi + \phi)$

Note z is a unit vector, only two independent parameters

S. Luo, Phys. Rev. A **77**, 042303 (2008)
M. Ali, ARPR, G. Alber, Phys. Rev. A **81**, 042105 (2010)
Sai Vinjanampathy and ARPR, J. Phys. A **45**, 095303 (2012)

A simple prescription for getting conditional density matrix for B when A a qubit, whatever B: qubit-qubit, qubit-qudit d > 2, N qubits

System AB with A a qubit



example of
d=4, or N=3

qubit-qubit,   qubit-qudit,   N qubits

Reduced density matrix for B: multiply each block by elements of $A_{\pm}$ and add blocks

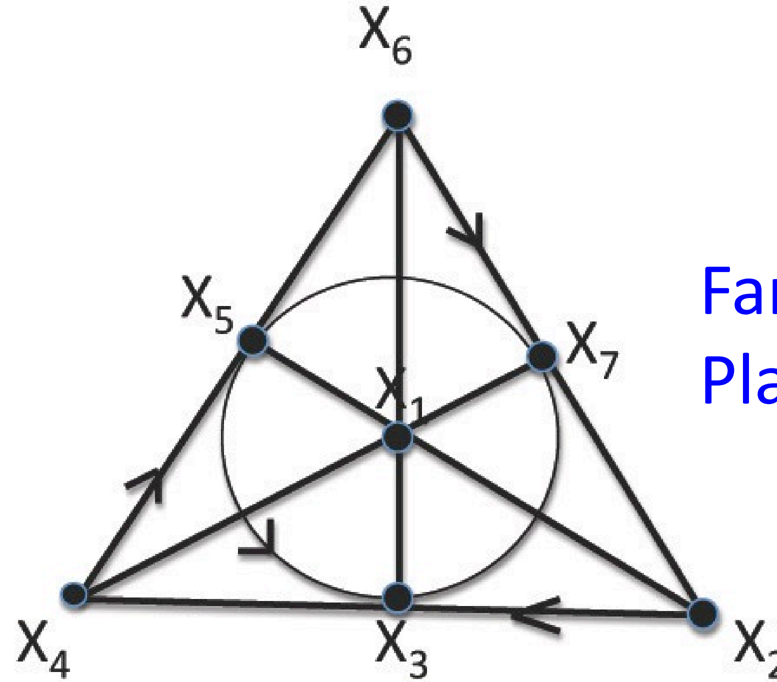
X states: only diagonal and anti-diagonal   Extended X states: blockwise

eigenvalue calculation analytical: quadratic or quartic roots

X-state density matrix

Bell states, Werner states, ...

$$\rho = \begin{pmatrix} \rho_{11} & 0 & 0 & \rho_{14} \\ 0 & \rho_{22} & \rho_{23} & 0 \\ 0 & \rho_{32} & \rho_{33} & 0 \\ \rho_{41} & 0 & 0 & \rho_{44} \end{pmatrix}$$



Fano Plane

3 real along diagonal, 2 complex on off-diagonal: 7 parameters

$$\rho = (I + \Sigma_i g_i X_i)/4$$

su(2) X u(1) X su(2) symmetry, 7 operators' algebra given by diagram

$$\rho = \frac{1}{4} I + \frac{1}{4} \begin{pmatrix} -g_4 & g_3 - ig_2 & g_6 - ig_5 & g_1 - g_7 \\ g_3 + ig_2 & g_4 & g_1 + g_7 & g_6 + ig_5 \\ g_6 + ig_5 & g_1 + g_7 & g_4 & g_3 + ig_2 \\ g_1 - g_7 & g_6 - ig_5 & g_3 - ig_2 & -g_4 \end{pmatrix}$$

N-qubit X-states: iterate:(su(2) X u(1) X su(2)) Xu(1)X (su(2) X u(1) X su(2))

ARPR, arXiv:0906.4716 & J.Phys.A **42**(2009);Phys. Rev. A**79**, 042323 (2009)

$$O_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \qquad O_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$O_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad O_5 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$O_6 = i \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \qquad O_7 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

$$O_8 = i \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \qquad O_9 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

plus O_{10} -- O_{16}

two spins qubit pair

I σ_i τ_i σ_i τ_j 15 operators SU(4)

O_X	σ_z	τ_z	$\sigma_z\tau_z$	σ_x	σ_y	$\sigma_x\tau_z$	$\sigma_y\tau_z$	τ_x	τ_y	$\sigma_z\tau_x$	$\sigma_z\tau_y$	$\sigma_x\tau_x$	$\sigma_y\tau_y$	$\sigma_x\tau_y$	$\sigma_y\tau_x$
σ_z	0	0	0	$i\sigma_y$	$-i\sigma_x$	$i\sigma_y\tau_z$	$-i\sigma_x\tau_z$	0	0	0	0	$i\sigma_y\tau_x$	$-i\sigma_x\tau_y$	$i\sigma_y\tau_y$	$-i\sigma_x\tau_x$
τ_z	0	0	0	0	0	0	0	$i\tau_y$	$-i\tau_x$	$i\sigma_z\tau_y$	$-i\sigma_z\tau_x$	$i\sigma_x\tau_y$	$-i\sigma_y\tau_x$	$-i\sigma_x\tau_x$	$i\sigma_y\tau_y$
$\sigma_z\tau_z$	0	0	0	$i\sigma_y\tau_z$	$-i\sigma_x\tau_z$	$i\sigma_y$	$-i\sigma_x$	$i\sigma_z\tau_y$	$-i\sigma_z\tau_x$	$i\tau_y$	$-i\tau_x$	0	0	0	0
σ_x	$-i\sigma_y$	0	$-i\sigma_y\tau_z$	0	$i\sigma_z$	0	$i\sigma_z\tau_z$	0	0	$-i\sigma_y\tau_x$	$-i\sigma_y\tau_y$	0	$i\sigma_z\tau_y$	0	$i\sigma_z\tau_x$
σ_y	$i\sigma_x$	0	$i\sigma_x\tau_z$	$-i\sigma_z$	0	0	$-i\sigma_z\tau_z$	0	0	0	$i\sigma_x\tau_y$	$-i\sigma_z\tau_x$	0	$-i\sigma_z\tau_y$	0
$\sigma_x\tau_z$	$-i\sigma_y\tau_z$	0	$-i\sigma_y$	0	$i\sigma_z\tau_z$	0	$i\sigma_z$	$i\sigma_x\tau_y$	$-i\sigma_x\tau_x$	0	0	$i\tau_y$	0	$-i\tau_x$	0
$\sigma_y\tau_z$	$i\sigma_x\tau_z$	0	$i\sigma_x$	$-i\sigma_z\tau_z$	0	$-i\sigma_z$	0	$i\sigma_y\tau_y$	$-i\sigma_y\tau_x$	0	0	0	$-i\tau_x$	0	$i\tau_y$
τ_x	0	$-i\tau_y$	$-i\sigma_z\tau_y$	0	0	$-i\sigma_x\tau_y$	$-i\sigma_y\tau_y$	0	$i\tau_z$	0	$i\sigma_z\tau_z$	0	$i\sigma_y\tau_z$	$i\sigma_x\tau_z$	0
τ_y	0	$i\tau_x$	$i\sigma_z\tau_x$	0	0	$i\sigma_x\tau_x$	$i\sigma_y\tau_x$	$-i\tau_z$	0	$-i\sigma_z\tau_z$	0	$-i\sigma_x\tau_z$	0	0	$-i\sigma_y\tau_z$
$\sigma_z\tau_x$	0	$-i\sigma_z\tau_y$	$-i\tau_y$	$i\sigma_y\tau_x$	$-i\sigma_x\tau_x$	0	0	0	$i\sigma_z\tau_z$	0	$i\tau_z$	$i\sigma_y$	0	0	$-i\sigma_x$
$\sigma_z\tau_y$	0	$i\sigma_z\tau_x$	$i\tau_x$	$i\sigma_y\tau_y$	$-i\sigma_x\tau_y$	0	0	$-i\sigma_z\tau_z$	0	$-i\tau_z$	0	0	$-i\sigma_x$	$i\sigma_y$	0
$\sigma_x\tau_x$	$-i\sigma_y\tau_x$	$-i\sigma_x\tau_y$	0	0	$i\sigma_z\tau_x$	$-i\tau_y$	0	0	$i\sigma_x\tau_z$	$-i\sigma_y$	0	0	0	$i\tau_z$	$i\sigma_z$
$\sigma_y\tau_y$	$i\sigma_x\tau_y$	$i\sigma_y\tau_x$	0	$-i\sigma_z\tau_y$	0	0	$i\tau_x$	$-i\sigma_y\tau_z$	0	0	$i\sigma_x$	0	0	$-i\sigma_z$	$-i\tau_z$
$\sigma_x\tau_y$	$-i\sigma_y\tau_y$	$i\sigma_x\tau_x$	0	0	$i\sigma_z\tau_y$	$i\tau_x$	0	$-i\sigma_x\tau_z$	0	0	$-i\sigma_y$	$-i\tau_z$	$i\sigma_z$	0	0
$\sigma_y\tau_x$	$i\sigma_x\tau_x$	$-i\sigma_y\tau_y$	0	$-i\sigma_z\tau_x$	0	0	$-i\tau_y$	0	$i\sigma_y\tau_z$	$i\sigma_x$	0	$-i\sigma_z$	$i\tau_z$	0	0

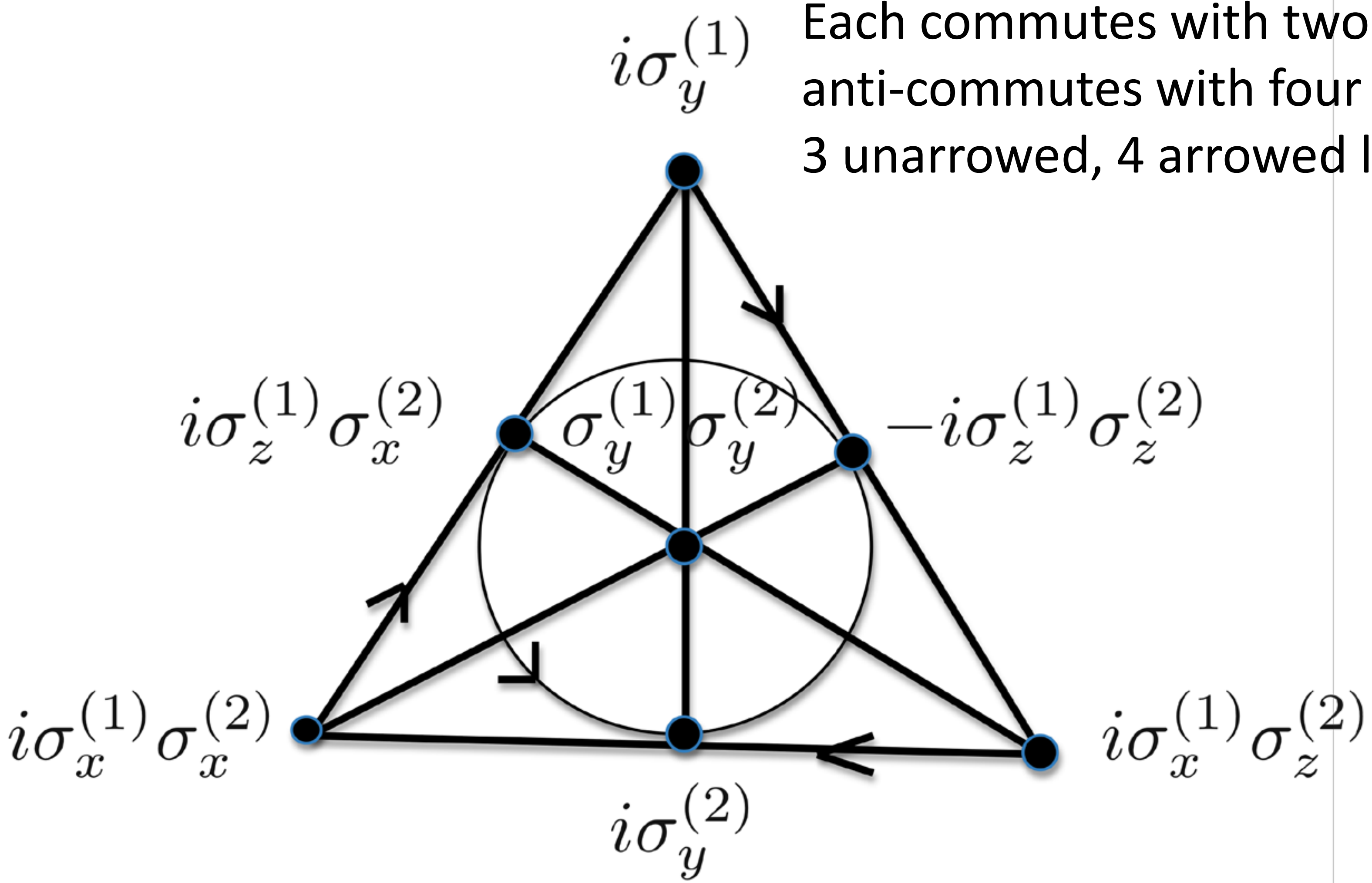
Table of Commutators

Each row has 7 zeroes, each commutes with 6 and anti with 8

Subgroup of SU(4) : SU(2) X U(1) X SU(2)

Seven generators; choose any one of 15, 6 others : 7 in all

Each commutes with two others, anti-commutes with four
3 unarrowed, 4 arrowed lines



Parameters in density matrices

Qudit – quDit : $d^2D^2 - 1$; qubit – qudit : $4d^2 - 1$; N qubits : $2^{2N} - 1$

N = 2 : 15, N = 3 : 63, N = 4 : 255, ...

X states : qubit - qudit : $4d - 1$; N qubits : $2^{N+1} - 1$

N = 2 : 7, N = 3 : 15, N = 4 : 31, ...

Extended X states : qubit – qudit : $8d - 1$ for d even $8d - 5$ for d odd

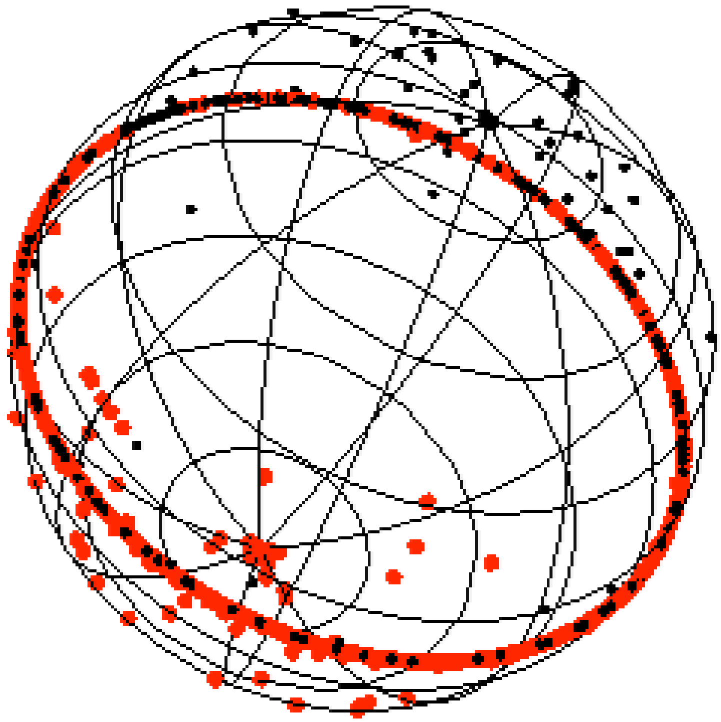
N qubits : $2^{N+2} - 1$; N = 2 : 15, N = 3 : 31, N =4 : 63, ...

Qubit-qubit : extended X states embrace all 15 of general density

N qubit discord

Two-parameter minimization θ, ϕ $\theta = 0, \pi/2$

10,000 randomly chosen two- and three- qubit density matrices



Sai Vinjanampathy and ARPR, arXiv 1106.4488 & J. Phys. A**45**, 095303('12)

Q. Chen etal 1102.0181 and X. Lu etal, Phys. Rev. A **83**, 012327 (2011)

Discord calculation when A is higher dimensional

counterpart of $A_{\pm}$: d projectors adding to unity

$$A_i = U\,\Pi_i\,U^\dagger$$

compact unitary matrix; e.g. only 2, not 3

Geometrical distance measure of discord:

$$D^{(2)}(\rho) = \min_{\chi \in \mathcal{C}} \|\rho - \chi\|^2$$

classical state B

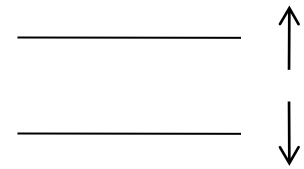
$$\chi = \sum_{i=1}^{d_A} p_i \Pi_i^{(A)} \otimes \rho_i^{(B)}$$

$$x_i \equiv \mathrm{tr}(\rho * \sigma_i \otimes I), y_i \equiv \mathrm{tr}(\rho * I \otimes \sigma_i) \quad T_{ij} \equiv \mathrm{tr}(\rho * \sigma_i \otimes \sigma_j)$$

$$D^{(2)}(\rho) = \frac{1}{4}(\|x\|^2 + \|T\|^2 - k_{\max}) \qquad xx^T + TT^T$$

Dakic, Vedral and Brukner, arXiv 1004.0190; Luo and Fu, PR A**82**, 034302

Spin ½, Two-level system, Qubit



SU(2) Symmetry: 3 parameters

$$\begin{pmatrix} c_+ \\ c_- \end{pmatrix} \qquad |c_+|^2 + |c_-|^2 = 1$$

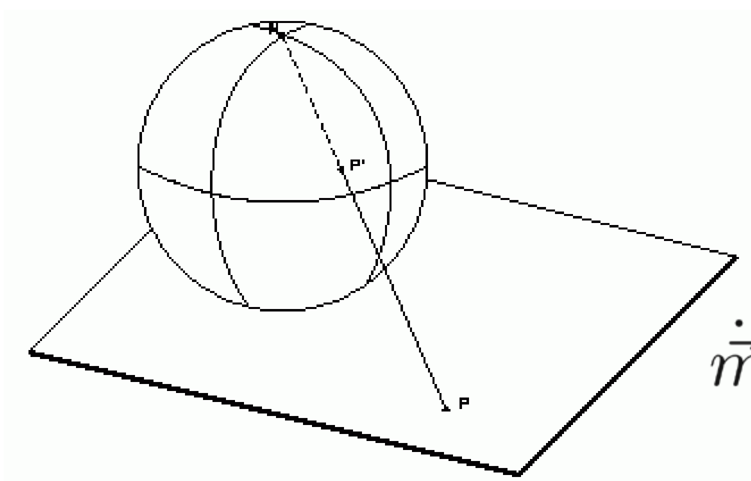
Evolution operator :

$$U(t) = e^{z(t)\sigma_+/2} e^{w^*(t)\sigma_-/2} e^{-i\mu(t)\sigma_z/2}$$

$$i\dot{\psi} = H(t)\psi(t), i\dot{\rho} = [H(t), \rho(t)]$$

$$w^* = -z^*/(1+|z|^2),\ e^{\mathrm{Im}\,\mu} = (1+|z|^2)$$

Inverse stereographic projection of z to Bloch sphere S^2

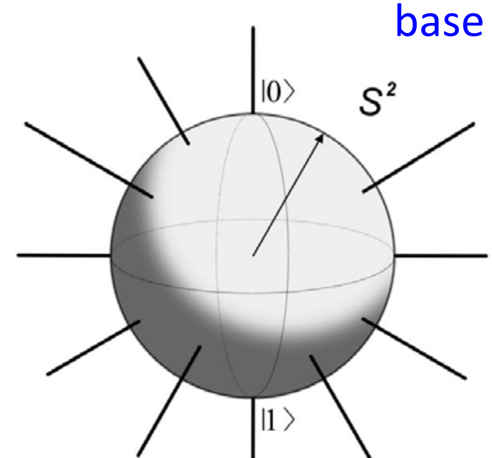


$\mathrm{Re}\ \mu$: U(1)

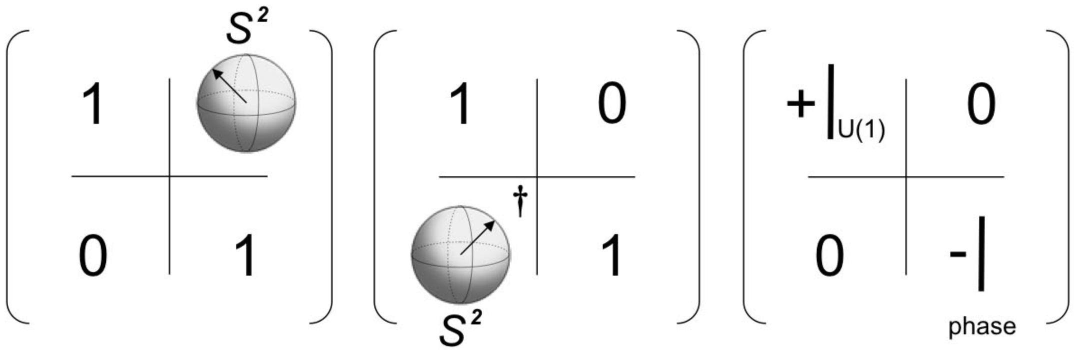
phase

$$\dot{\vec{m}} = -2\vec{B} \times \vec{m}$$

base



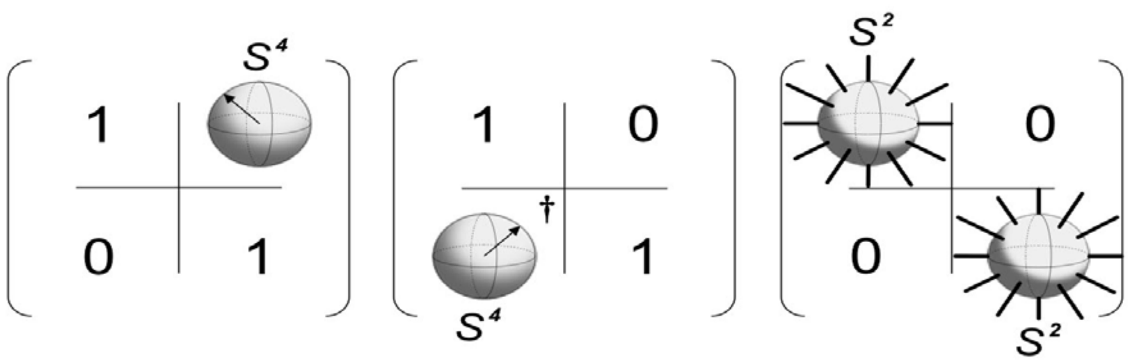
U =



$$SU(2) : S^2 \times U(1)$$

Bloch sphere base space X fiber, a single phase

U =



$$H = \begin{pmatrix} (N-n)\times(N-n) & (N-n)\times n \\ n\times(N-n) & n\times n \end{pmatrix}$$

$$\downarrow$$

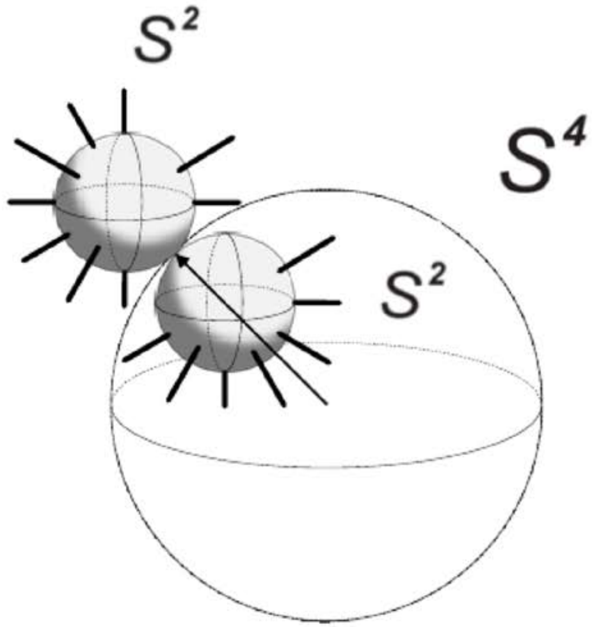
$$H_{eff} = \begin{pmatrix} (N-n)\times(N-n) & 0 \\ 0 & n\times n \end{pmatrix}$$

SU(4):Four-sphere base X fiber of two SU(2) manifolds

4-level systems, Two spins/qubits: SU(4) Symmetry,15 paramet

Sub-algebra of so(5) with 10 parameters

D.B.Uskov and ARPR,
Phys. Rev. A **74**,
030304 (R) (2006)
and **78**, 022331 (2008)



Sai Vinjanampathy and ARPR,
J. Phys. A **42**, 425303 (2009)

A four-sphere, the analog of Bloch two-sphere: 4 parameters.
At each point on it, not a single phase as for single spin but two "spiked Bloch spheres" for a total of six parameters.