ECE 595: Machine Learning I
Lecture 4.2: Intro to Optimization - Convexity

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Outline

Mathematical Background

- Lecture 4: Intro to Optimization
- Lecture 5: Gradient Descent

Lecture 4: Intro to Optimization

- Unconstrained Optimization
  - First Order Optimality
  - Second Order Optimality
- Convexity
  - What is convexity?
  - Convex optimization
- Constrained Optimization
  - Lagrangian
  - Examples

Most Optimization Problems are Not Easy

Minimize the log-sum-exp function:

$$f(\boldsymbol{x}) = \log\left(\sum_{i=1}^{m} \exp(\boldsymbol{a}_i^T \boldsymbol{x} + b_i)\right)$$

- Gradient is (exercise)

$$\nabla f(\boldsymbol{x}^*) = \frac{1}{\sum_{j=1}^{m} \exp(\boldsymbol{a}_j^T \boldsymbol{x}^* + b_j)} \sum_{i=1}^{m} \exp(\boldsymbol{a}_i^T \boldsymbol{x}^* + b_i)\boldsymbol{a}_i.$$

- Non-linear equation. No closed-form solution.
- Need iterative algorithms, e.g., gradient descent.
- Or off-the-shelf optimization solver, e.g., `CVX`.

CVX Demonstration

- Disciplined optimization: It translates the problem for you.
- Developed by S. Boyd and colleagues (Stanford).
- E.g., Minimize $f(\boldsymbol{x}) = \log\left(\sum_{i=1}^{n} \exp(\boldsymbol{a}_i^T \boldsymbol{x} + b_i)\right) + \lambda\|\boldsymbol{x}\|^2$.

```
import cvxpy as cp
import numpy as np

n = 100
d = 3
A = np.random.randn(n, d)
b = np.random.randn(n)
lambda_ = 0.1

x = cp.Variable(d)
objective = cp.Minimize(cp.log_sum_exp(A*x - b) + lambda_*cp.sum_squares(x))
constraints = []
prob = cp.Problem(objective, constraints)

optimal_objective_value = prob.solve()
print(optimal_objective_value)
print(x.value)
```

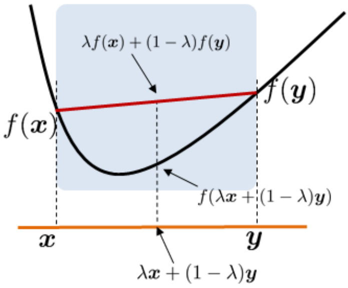
Convex Function

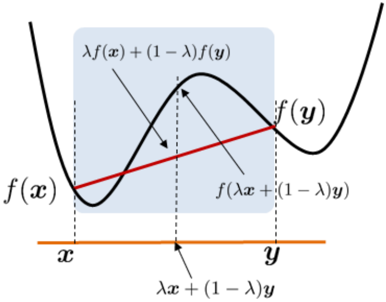
Definition

Let $\boldsymbol{x} \in \mathcal{X}$ and $\boldsymbol{y} \in \mathcal{X}$. Let $0 \leq \lambda \leq 1$. A function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is **convex** over $\mathcal{X}$ if

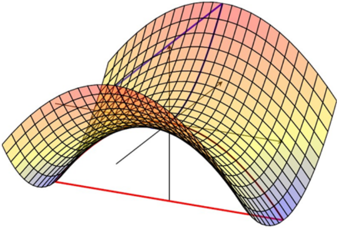
$$f(\lambda \boldsymbol{x} + (1-\lambda)\boldsymbol{y}) \;\leq\; \lambda f(\boldsymbol{x}) + (1-\lambda) f(\boldsymbol{y}).$$

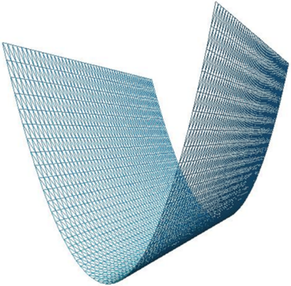
The function is called strictly convex if "$\leq$" is replaced by "$<$".

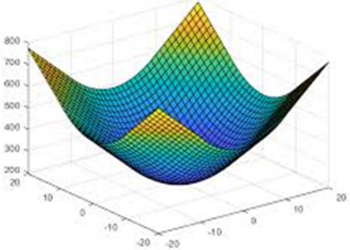


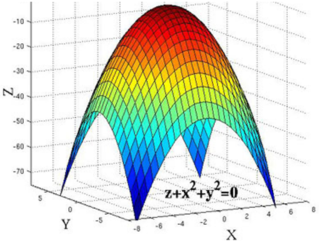


Example: Which one is convex?









Verifying Convexity

Any of the following conditions is **necessary** and **sufficient** for convexity:

1. By definition:

$$f(\lambda \boldsymbol{x} + (1-\lambda)\boldsymbol{y}) \leq \lambda f(\boldsymbol{x}) + (1-\lambda) f(\boldsymbol{y}).$$

   - Function value is lower than the line.

2. First Order Convexity:

$$f(\boldsymbol{y}) \geq f(\boldsymbol{x}) + \nabla f(\boldsymbol{x})^T (\boldsymbol{y} - \boldsymbol{x}), \qquad \forall \boldsymbol{x}, \boldsymbol{y} \in \mathcal{X}.$$

   - Tangent line is always lower than the function

3. Second Order Convexity: f is convex over $\mathcal{X}$ if and only if

$$\nabla^2 f(\boldsymbol{x}) \succeq 0 \quad \forall \boldsymbol{x} \in \mathcal{X}.$$

   - Curvature is positive.

Tangent Line Condition Illustrated