ECE595 / STAT598: Machine Learning I
Lecture 17.2: Perceptron 2 - Optimality

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Outline

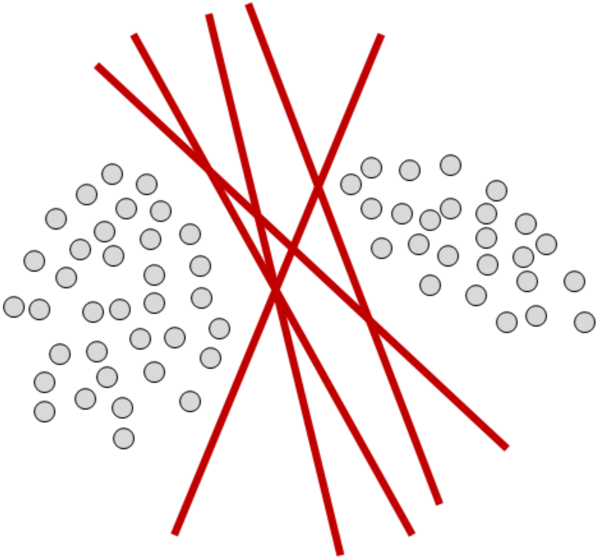
Discriminative Approaches

- Lecture 16 Perceptron 1: Definition and Basic Concepts
- Lecture 17 Perceptron 2: Algorithm and Property
- Lecture 18 Multi-Layer Perceptron: Back Propagation

This lecture: Perceptron 2

- Perceptron Algorithm
  - Loss Function
  - Algorithm
- Optimality
  - Uniqueness
  - Batch and Online Mode
- Convergence
  - Main Results
  - Implication

Non-uniqueness of Global Minimizer



Optimality of Perceptron Algorithm

- Let **perceptron algorithm** output

$$\boldsymbol{\theta}^*_{\text{perceptron}} = \text{Perceptron Algorithm}(\{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N\}).$$

- Let **ideal** solution

$$\boldsymbol{\theta}^*_{\text{hard}} = \underset{\boldsymbol{\theta}}{\text{argmin}} \; J_{\text{hard}}(\boldsymbol{\theta}).$$

  That means

$$J_{\text{hard}}(\boldsymbol{\theta}^*_{\text{hard}}) \leq J_{\text{hard}}(\boldsymbol{\theta}), \quad \forall \boldsymbol{\theta}.$$

- If the two classes are linearly separable, then $\boldsymbol{\theta}^*_{\text{perceptron}}$ is a global minimizer:

$$J_{\text{hard}}(\boldsymbol{\theta}^*_{\text{perceptron}}) \leq J_{\text{hard}}(\boldsymbol{\theta}), \quad \forall \boldsymbol{\theta}.$$

  and

$$J_{\text{hard}}(\boldsymbol{\theta}^*_{\text{perceptron}}) = J_{\text{hard}}(\boldsymbol{\theta}^*_{\text{hard}}) = 0.$$

Uniqueness of Perceptron Solution

- If $\boldsymbol{\theta}^*$ minimizes $J_{\text{hard}}(\boldsymbol{\theta}^*)$, then $\alpha\boldsymbol{\theta}^*$ for some constant $\alpha > 0$ also minimizes $J_{\text{hard}}(\boldsymbol{\theta}^*)$.
- This is because

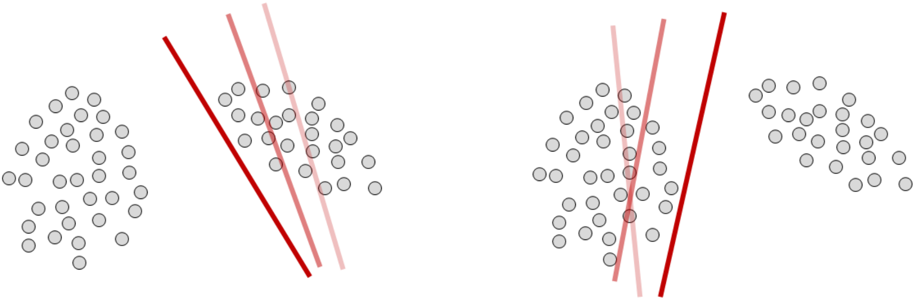
$$\begin{aligned} g_{\alpha\boldsymbol{\theta}}(\boldsymbol{x}) &= (\alpha\boldsymbol{w})^T\boldsymbol{x} + (\alpha w_0) \\ &= \alpha(\boldsymbol{w}^T\boldsymbol{x} + w_0). \end{aligned}$$

- If $g_{\boldsymbol{\theta}}(\boldsymbol{x}) > 0$, then $g_{\alpha\boldsymbol{\theta}}(\boldsymbol{x}) > 0$. So if $h_{\boldsymbol{\theta}}(\boldsymbol{x}) = +1$, then $h_{\alpha\boldsymbol{\theta}}(\boldsymbol{x}) = +1$.
- If $g_{\boldsymbol{\theta}}(\boldsymbol{x}) < 0$, then $g_{\alpha\boldsymbol{\theta}}(\boldsymbol{x}) < 0$. So if $h_{\boldsymbol{\theta}}(\boldsymbol{x}) = -1$, then $h_{\alpha\boldsymbol{\theta}}(\boldsymbol{x}) = -1$.
- The sign of $\boldsymbol{w}^T\boldsymbol{x} + w_0$ is unchanged as long as $\alpha > 0$.

$$\begin{aligned} J_{\text{hard}}(\boldsymbol{\theta}^*) &= \sum_{j=1}^{N} \max\left\{ -y_j h_{\boldsymbol{\theta}^*}(\boldsymbol{x}_j), 0 \right\} \\ &= \sum_{j=1}^{N} \max\left\{ -y_j h_{\alpha\boldsymbol{\theta}^*}(\boldsymbol{x}_j), 0 \right\} = J_{\text{hard}}(\alpha\boldsymbol{\theta}^*) \end{aligned}$$

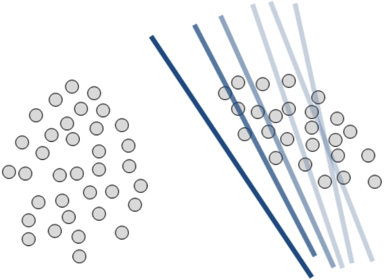
Factors for Uniqueness

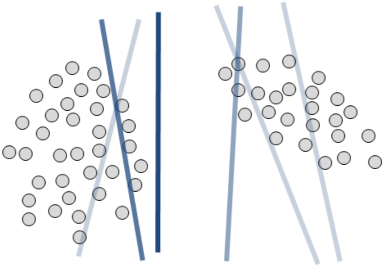
- **Initialization**
- Start at a different location, end on a different location
- You still converge, but no longer unique solution
- $\mathcal{M}_k$ changes



Factors for Uniqueness

- **Step Size**
- Too large step: oscillate
- Too small step: slow movement
- Terminates as long as no misclassification





Batch vs Online Mode

- **Batch mode**

$$\begin{bmatrix} \boldsymbol{w}^{(k+1)} \\ w_0^{(k+1)} \end{bmatrix} = \begin{bmatrix} \boldsymbol{w}^{(k)} \\ w_0^{(k)} \end{bmatrix} + \alpha_k \sum_{j \in \mathcal{M}_k} \begin{bmatrix} y_j \boldsymbol{x}_j \\ y_j \end{bmatrix}.$$

  Update via the average of misclassified samples

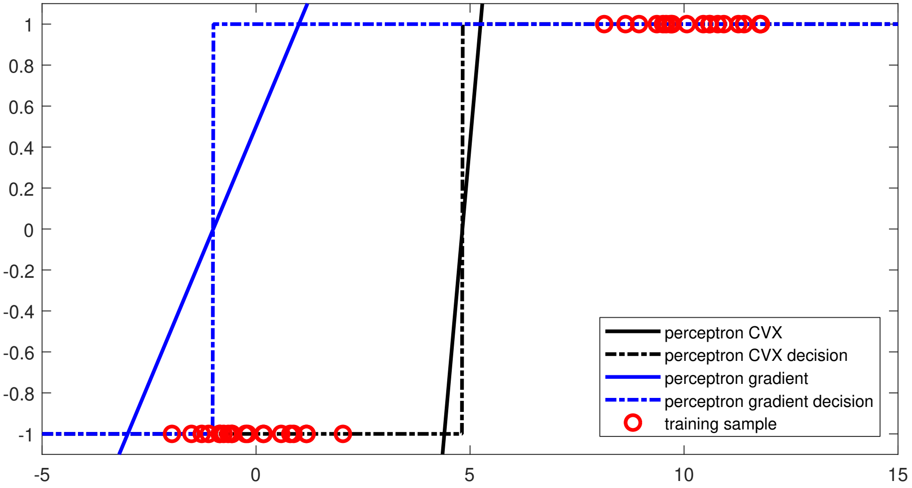
- **Online mode**

$$\begin{bmatrix} \boldsymbol{w}^{(k+1)} \\ w_0^{(k+1)} \end{bmatrix} = \begin{bmatrix} \boldsymbol{w}^{(k)} \\ w_0^{(k)} \end{bmatrix} + \alpha_k \begin{bmatrix} y_j \boldsymbol{x}_j \\ y_j \end{bmatrix},$$

  Update via a single misclassified sample

- j is a sample randomly picked from $\mathcal{M}_k$.
- Stochastic gradient descent.

Online Mode



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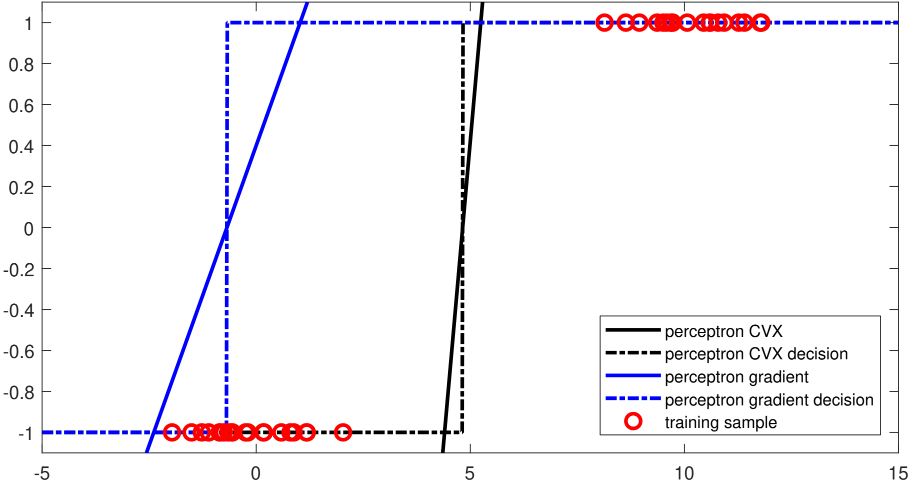
perceptron CVX
perceptron CVX decision
perceptron gradient
perceptron gradient decision
training sample

Online Mode



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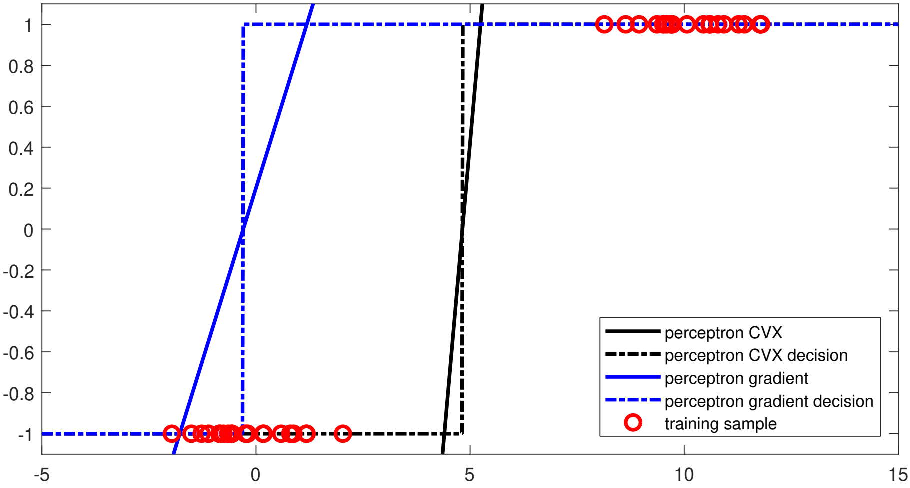

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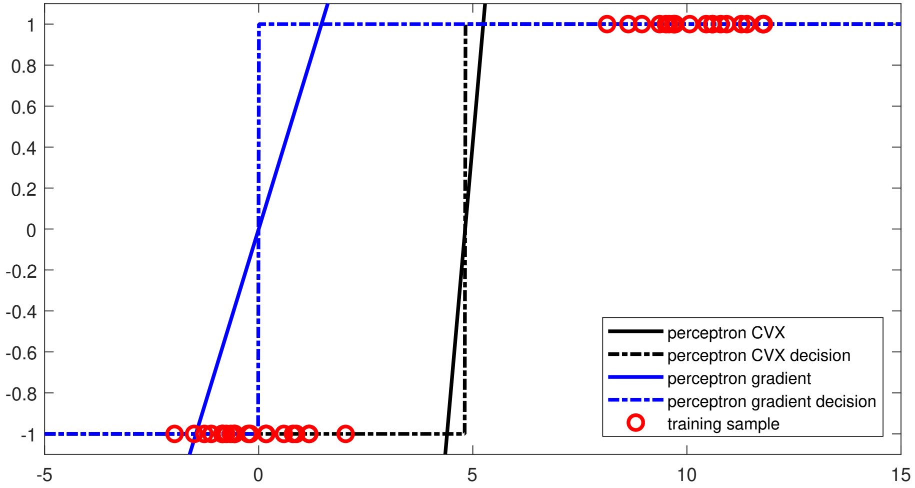

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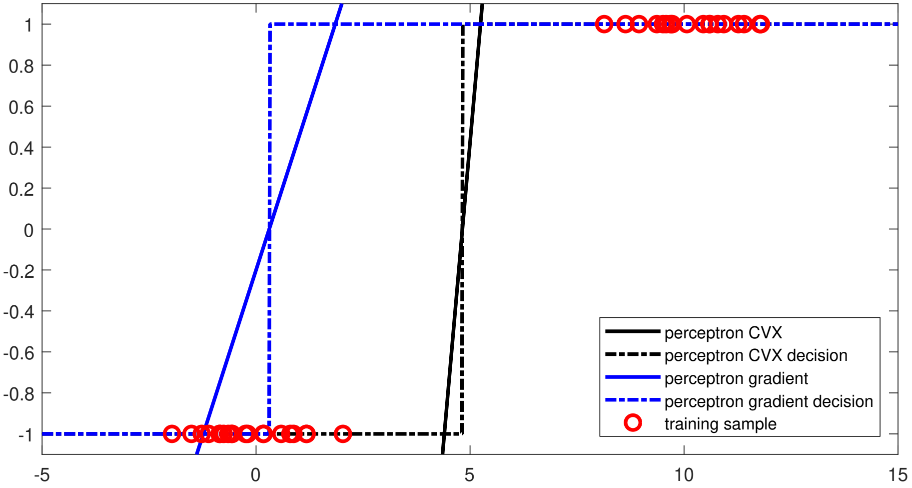
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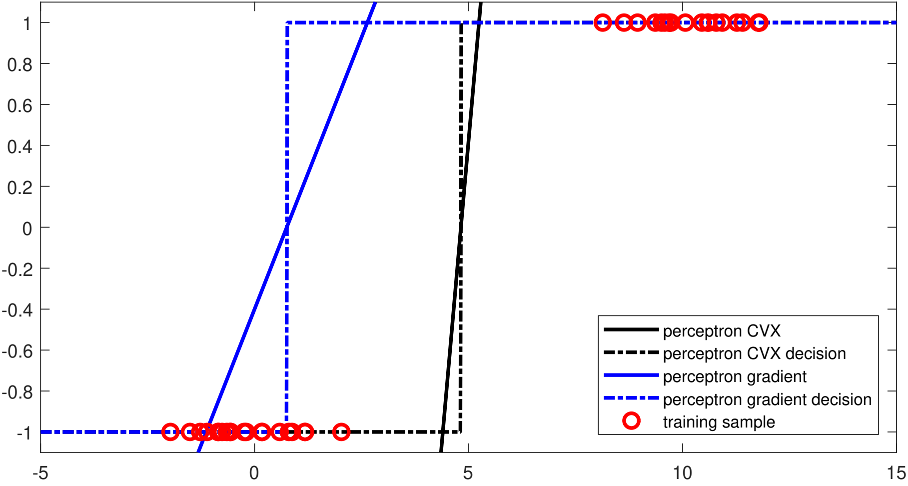

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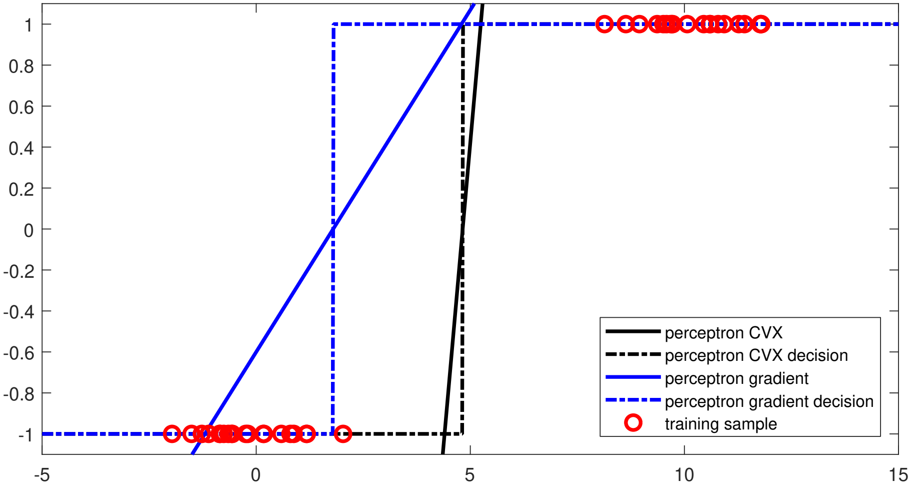

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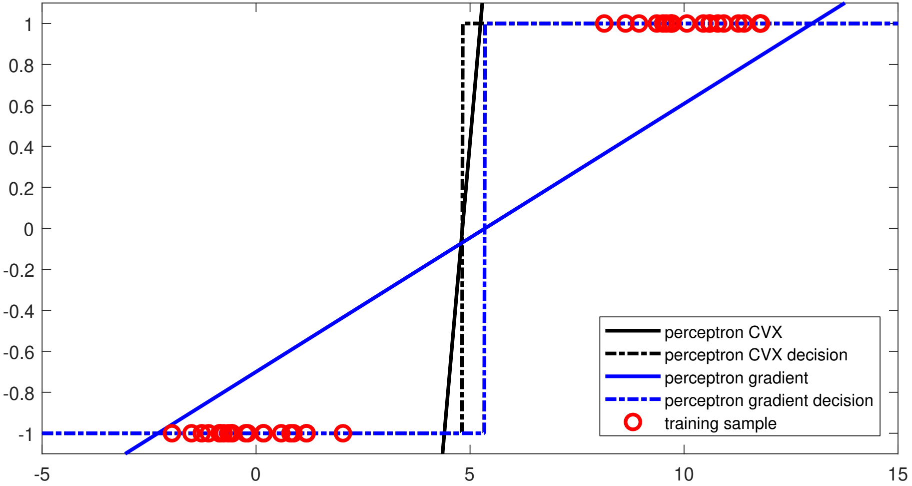

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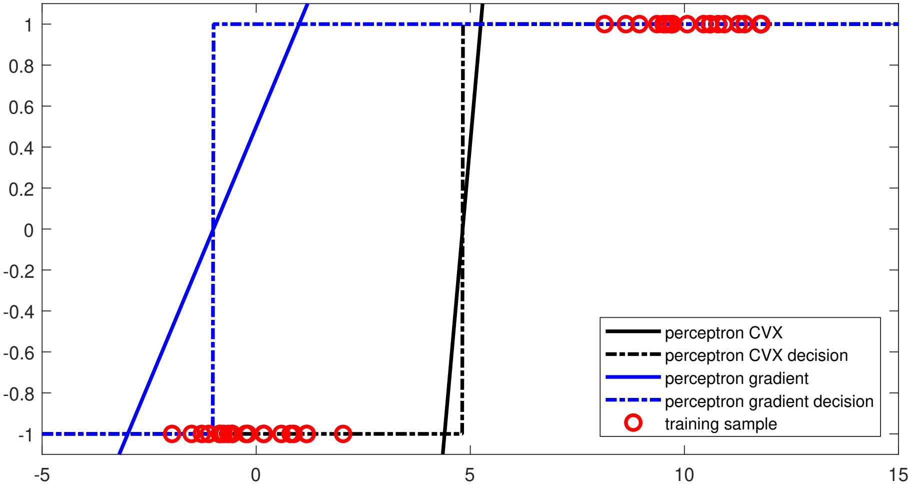
Batch Mode



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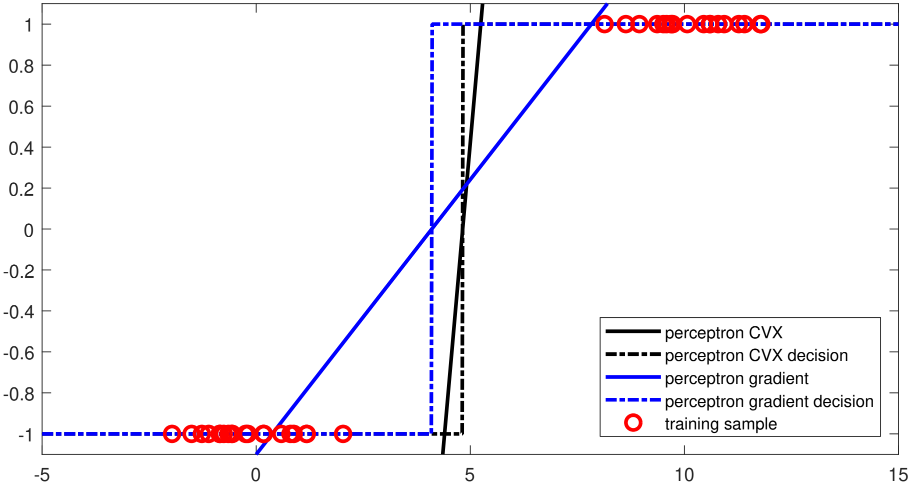

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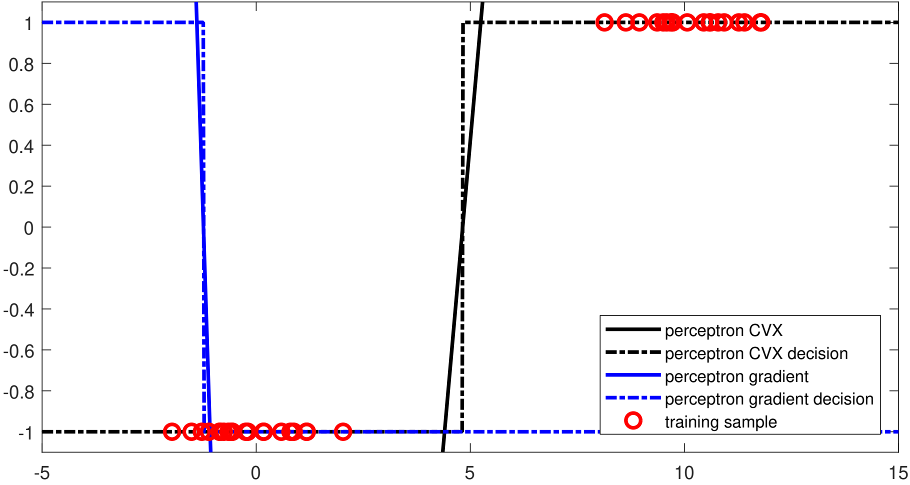
Step Size

Batch mode: Step size too large.



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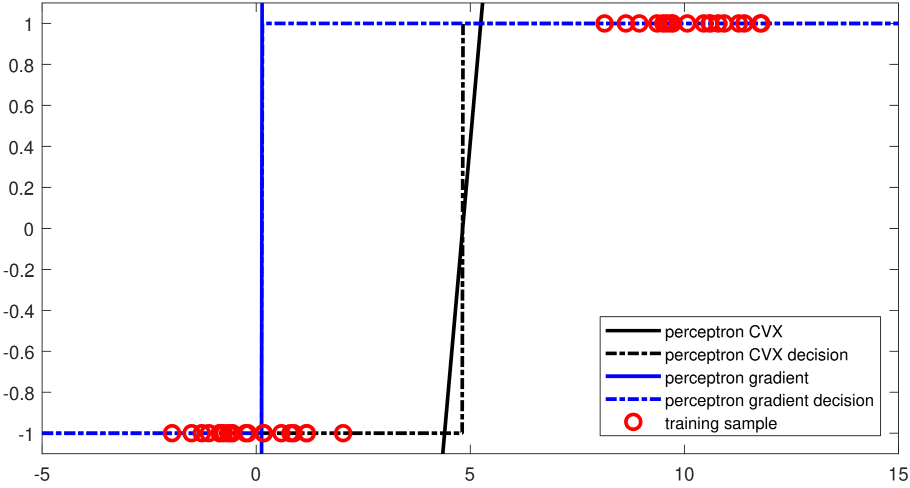

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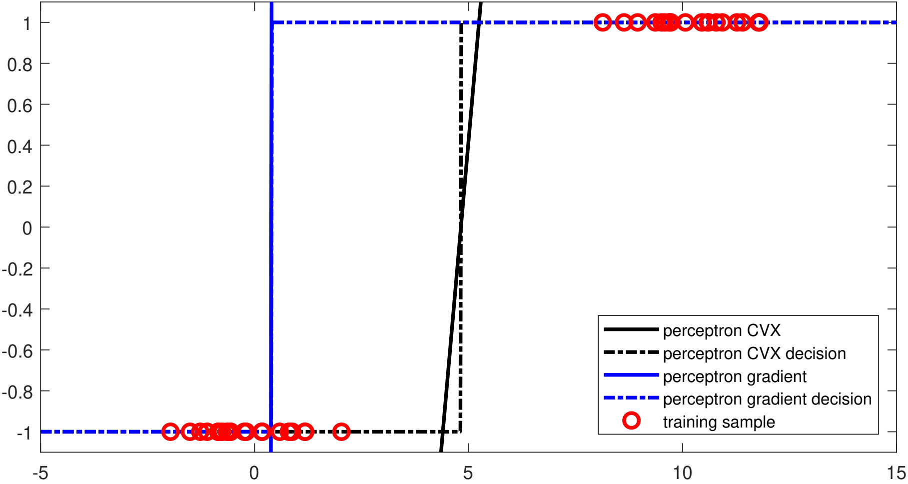

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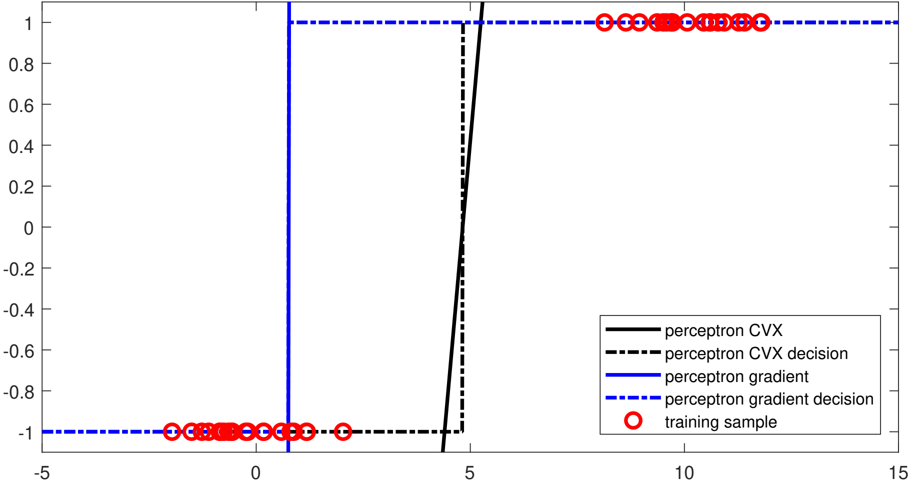

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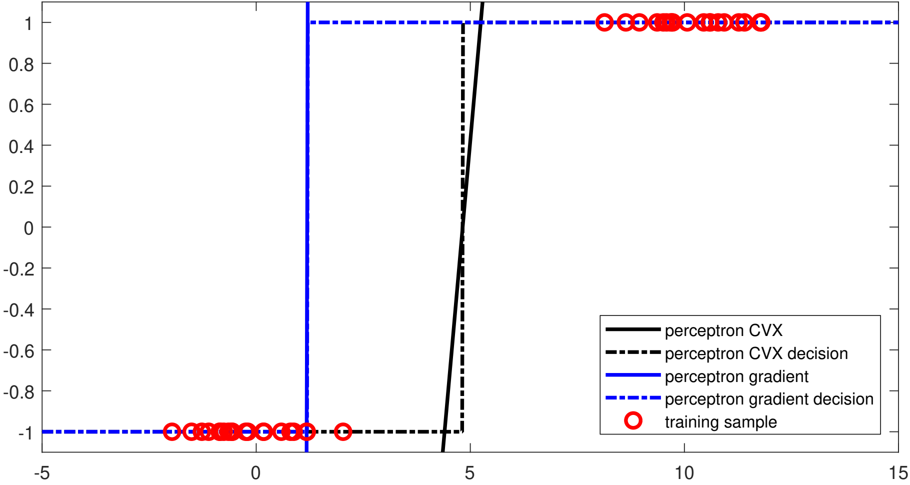

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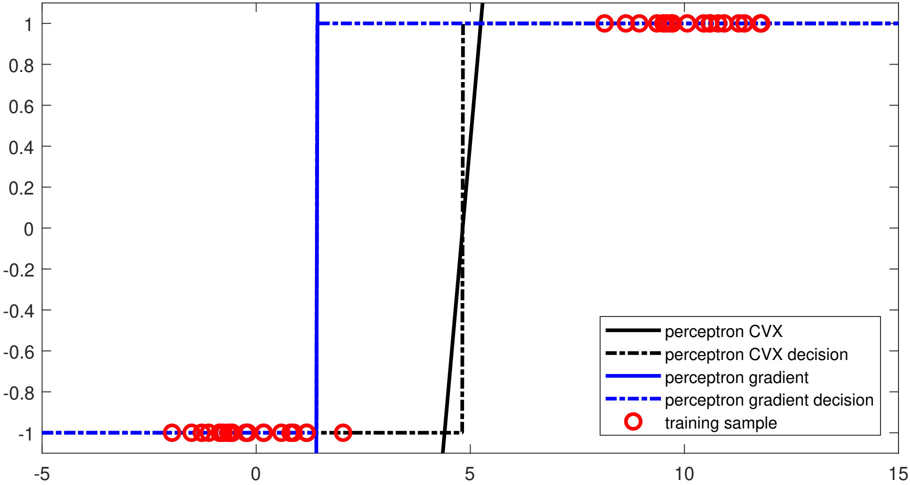

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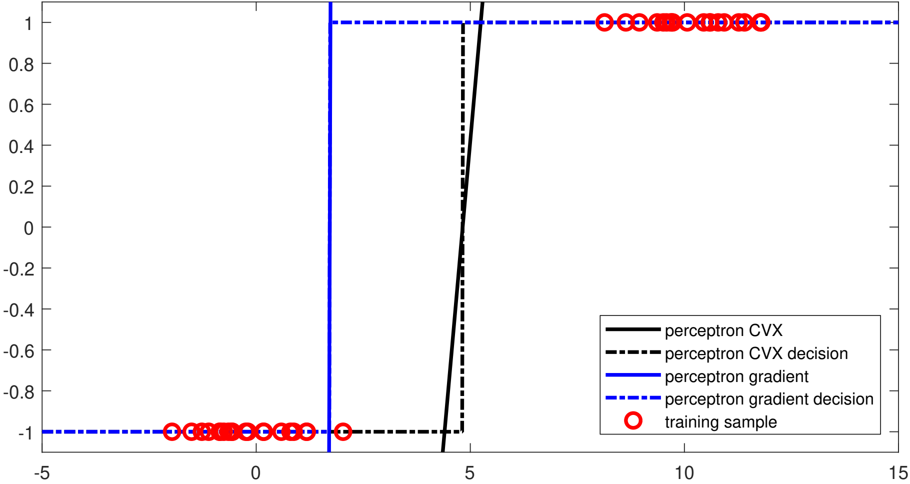

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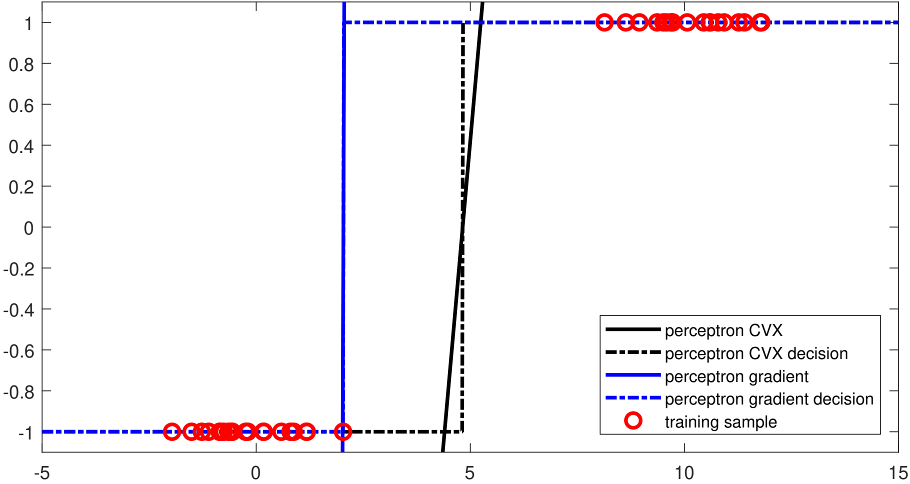
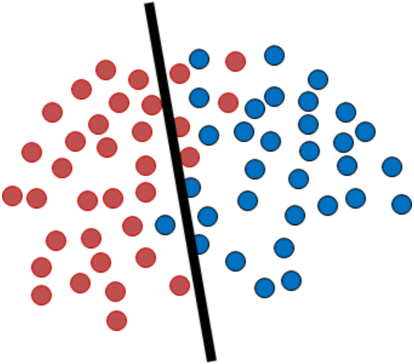


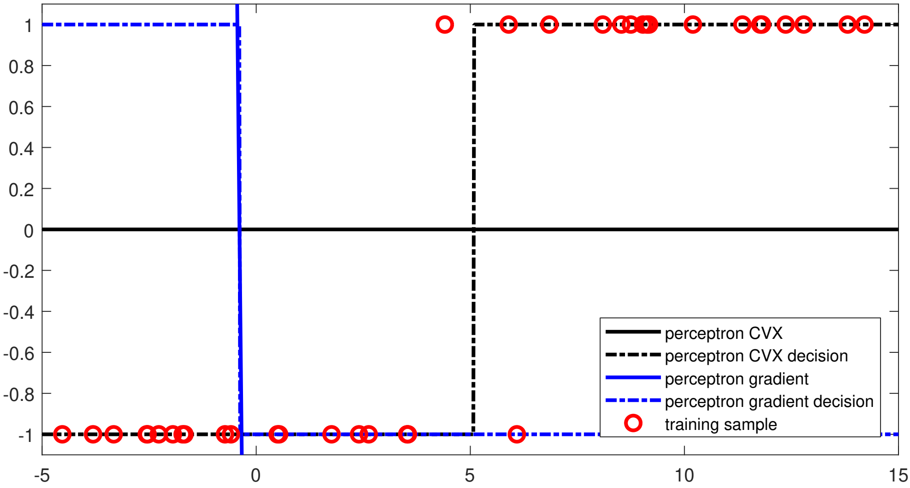
Linearly Not Separable



- No separating hyperplane
- CVX will still find you a solution
- But loss is no longer zero
- Perceptron algorithm will not converge

Linearly Not Separable

If the two classes are overlapping



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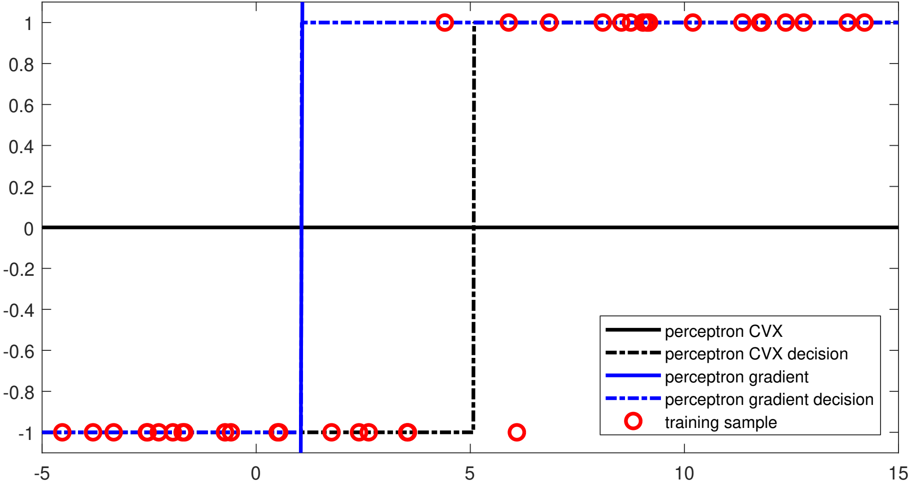

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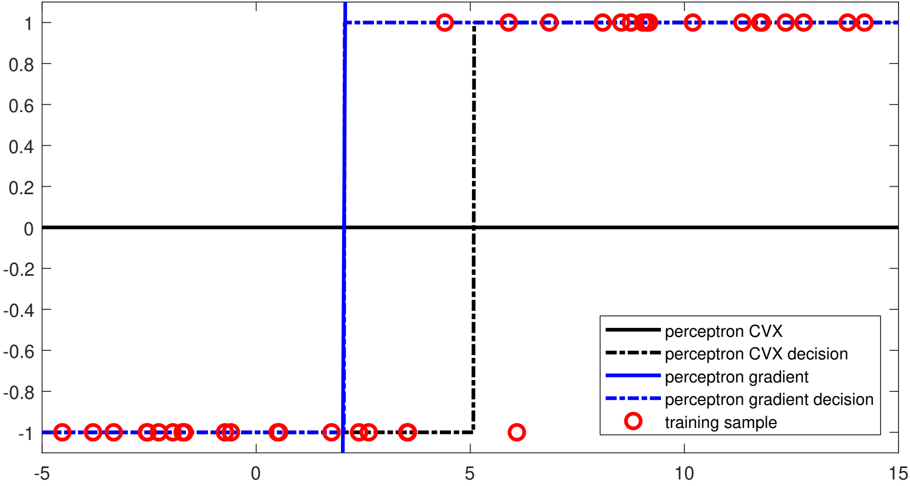

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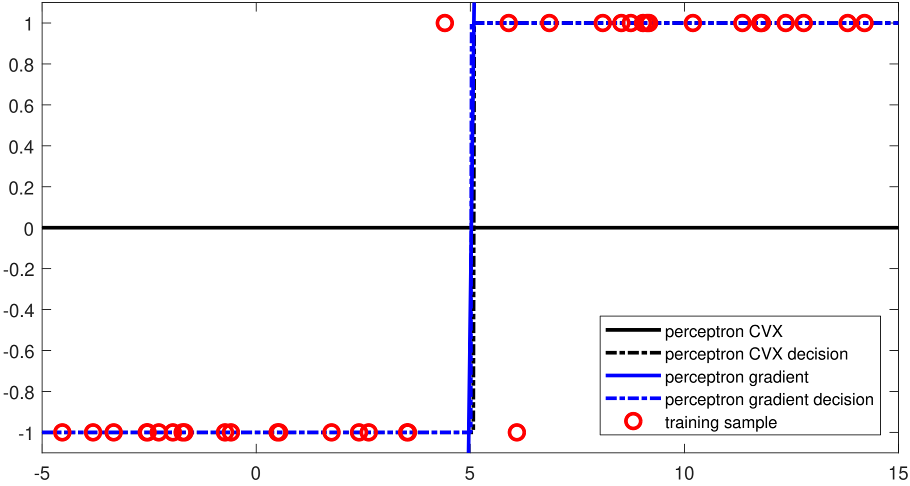

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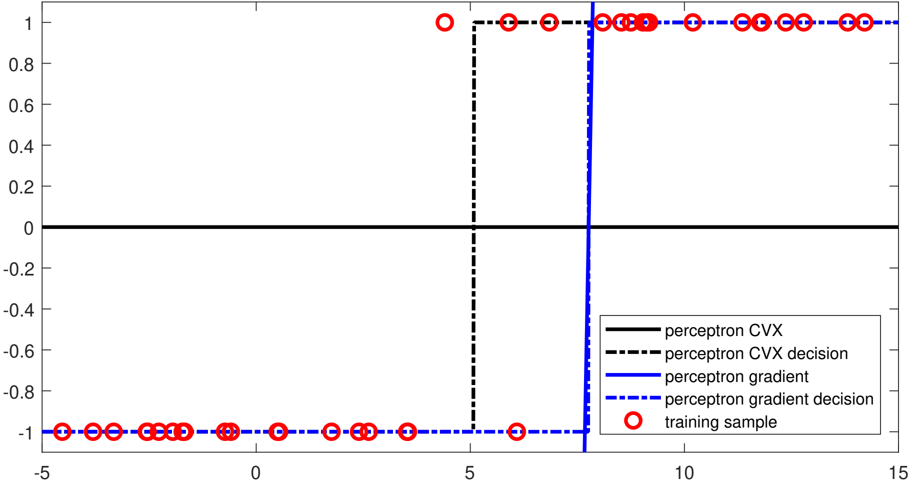

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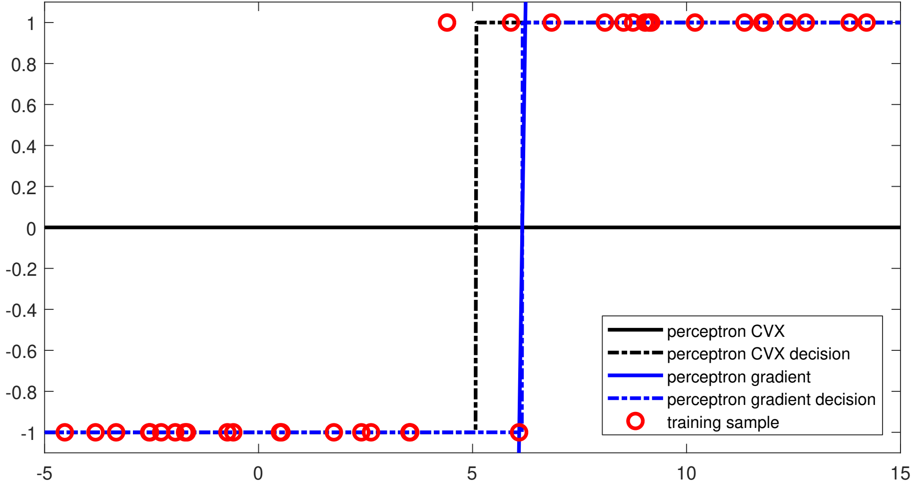

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