ECE595 / STAT598: Machine Learning I
Lecture 27:2: Example of VC Dimension

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Stanley Chan

School of Electrical and Computer Engineering
Purdue University

Outline

- Lecture 25 Generalization
- Lecture 26 Growth Function
- Lecture 27 VC Dimension

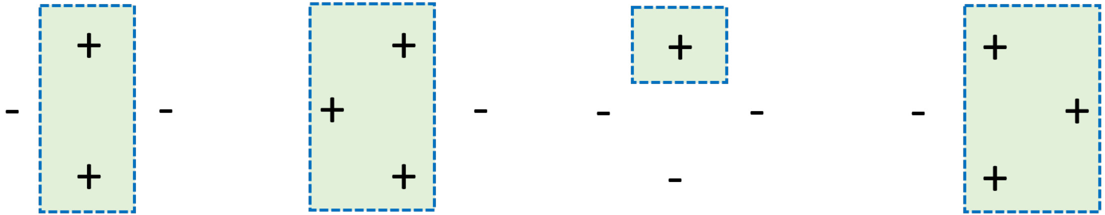
Today's Lecture:

- From Dichotomy to Shattering
  - Review of dichotomy
  - The Concept of Shattering
  - VC Dimension
- Example of VC Dimension
  - Rectangle Classifier
  - Perceptron Algorithm
  - Two Cases

Example: Rectangle

What is the VC Dimension of a 2D classifier with a rectangle shape?

- You can try putting 4 data points in whatever way.
- There will be 16 possible configurations.
- You can show that the rectangle classifier can shatter all these 16 points
- If you do 5 data points, then not possible. (Put one negative in the interior, and four positive at the boundary.)
- So VC dimension is 4.



VC Dimension of a Perceptron

Theorem (VC Dimension of a Perceptron)

Consider the input space $\mathcal{X} = \mathbb{R}^d \cup \{1\}$, *i.e.,* ($\boldsymbol{x} = [1, x_1, \ldots, x_d]^T$). *The VC dimension of a perceptron is*

$$d_{\mathrm{VC}} = d + 1.$$

- The "+1" comes from the bias term (w_0 if you recall)
- So a linear classifier is "no more complicated" than $d+1$
- The best it can shatter is $d+1$ in a d-dimensional space
- E.g., If $d = 2$, then $d_{\mathrm{VC}} = 3$

Why?

- We claim $d_{\mathrm{VC}} \geq d+1$ and $d_{\mathrm{VC}} \leq d+1$
- $d_{\mathrm{VC}} \geq d+1$:

$$\mathcal{H} \text{ can shatter } \textbf{at least } d+1 \text{ points}$$

- It may shatter more, or it may not shatter more. We don't know by just looking at this statement
- $d_{\mathrm{VC}} \leq d+1$:

$$\mathcal{H} \text{ cannot shatter } \textbf{more than } d+1 \text{ points}$$

- So with $d_{\mathrm{VC}} \geq d+1$, we show that $d_{\mathrm{VC}} = d+1$

$d_{\mathrm{VC}} \geq d+1$

- Goal: Show that there is at least one configuration of $d+1$ points that can be shattered by $\mathcal{H}$
- Think about the 2D case: Put the three points anywhere not on the same line
- Choose

$$\boldsymbol{x}_n = [1,\ 0, \ldots,\ 1,\ \ldots,\ 0]^T.$$

- Linear classifier: $\mathrm{sign}(\boldsymbol{w}^T \boldsymbol{x}_n) = y_n$.
- For all $d+1$ data points, we have

$$\mathrm{sign}\left(\begin{bmatrix} 1 & 0 & 0 & \ldots & 0 \\ 1 & 1 & 0 & \ldots & 0 \\ 1 & 0 & 1 & & 0 \\ & & & \ddots & 0 \\ 1 & 0 & \ldots & 0 & 1 \end{bmatrix} \begin{bmatrix} w_0 \\ w_1 \\ \vdots \\ w_d \end{bmatrix}\right) = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{d+1} \end{bmatrix} = \begin{bmatrix} \pm 1 \\ \pm 1 \\ \vdots \\ \pm 1 \end{bmatrix}$$

$d_{\mathrm{VC}} \geq d + 1$

- We can remove the sign because we are trying to find **one** configuration of points that can be shattered.

$$
\begin{bmatrix}
1 & 0 & 0 & \dots & 0 \\
1 & 1 & 0 & \dots & 0 \\
1 & 0 & 1 & & 0 \\
 & & & \ddots & 0 \\
1 & 0 & \dots & 0 & 1
\end{bmatrix}
\begin{bmatrix} w_0 \\ w_1 \\ \vdots \\ w_d \end{bmatrix}
=
\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{d+1} \end{bmatrix}
=
\begin{bmatrix} \pm 1 \\ \pm 1 \\ \vdots \\ \pm 1 \end{bmatrix}
$$

- We are only interested in whether the problem **solvable**
- So we just need to see if we can ever find a $\boldsymbol{w}$ that shatters
- If there exists at least one $\boldsymbol{w}$ that makes all ± 1 correct, then $\mathcal{H}$ can shatter (if you use that particular $\boldsymbol{w}$)
- So is this $(d+1) \times (d+1)$ system invertible?
- Yes. It is. So $\mathcal{H}$ can shatter at least $d+1$ points

$d_{\mathrm{VC}} \leq d + 1$

- Can we shatter more than $d+1$ points?
- No.
- You only have $d+1$ variables
- If you have $d+2$ equations, then one equation will be either redundant or contradictory
- If redundant, you can ignore it because it is not the worst case
- If contradictory, then you cannot solve the system of linear equation
- So we cannot shatter more than $d+1$ points
- You can always construct a nasty $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_{d+1}$ to cause contradiction

$d_{\mathrm{VC}} \leq d+1$

- You give me $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_{d+1}, \boldsymbol{x}_{d+2}$
- I can always write $\boldsymbol{x}_{d+2}$ as

$$\boldsymbol{x}_{d+2} = \sum_{i=1}^{d+1} a_i \boldsymbol{x}_i$$

- Not all a_i's are zero. Otherwise it will be trivial.
- My job: Construct a dichotomy which cannot be shattered by any h.
- Here is a dichotomy.
- $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_{d+1}$ get $y_i = \mathrm{sign}(a_i)$.
- $\boldsymbol{x}_{d+2}$ gets $y_{d+2} = -1$.

$d_{\mathrm{VC}} \leq d + 1$

- Then
$$\mathbf{w}^T \mathbf{x}_{d+2} = \sum_{i=1}^{d+1} a_i \mathbf{w}^T \mathbf{x}_i.$$
- Perceptron: $y_i = \text{sign}(\mathbf{w}^T \mathbf{x}_i)$.
- By our design, $y_i = \text{sign}(a_i)$.
- So $a_i \mathbf{w}^T \mathbf{x}_i > 0$
- This forces
$$\sum_{i=1}^{d+1} a_i \mathbf{w}^T \mathbf{x}_i > 0.$$
- So $y_{d+2} = \text{sign}(\mathbf{w}^T \mathbf{x}_{d+2}) = +1$, contradiction.
- So we found a dichotomy which cannot be shattered by any h.

Summary of the Examples

- $\mathcal{H}$ is positive ray: $m_{\mathcal{H}}(N) = N + 1$.
  - If $N = 1$, then $m_{\mathcal{H}}(1) = 2$
  - If $N = 2$, then $m_{\mathcal{H}}(2) = 3$
  - So $d_{\mathrm{VC}} = 1$
- $\mathcal{H}$ is positive interval: $m_{\mathcal{H}}(N) = \frac{N^2}{2} + \frac{N}{2} + 1$.
  - If $N = 2$, then $m_{\mathcal{H}}(2) = 4$
  - If $N = 4$, then $m_{\mathcal{H}}(4) = 5$
  - So $d_{\mathrm{VC}} = 2$
- $\mathcal{H}$ is perceptron in d-dimensional space
  - Just showed
  - $d_{\mathrm{VC}} = d + 1$
- $\mathcal{H}$ is convex set: $m_{\mathcal{H}}(N) = 2^N$
  - No matter which N we choose, we always have $m_{\mathcal{H}}(N) = 2^N$
  - So $d_{\mathrm{VC}} = \infty$
  - The model is as complex as it can be

Reading List

- Yasar Abu-Mostafa, Learning from Data, chapter 2.1
- Mehrya Mohri, Foundations of Machine Learning, Chapter 3.2
- Stanford Note http://cs229.stanford.edu/notes/cs229-notes4.pdf

Appendix

Radon Theorem

The perceptron example we showed in this lecture can be proved using Radon's theorem.

Theorem (Radon's Theorem)

Any set $\mathcal{X}$ of $d+2$ data points in $\mathbb{R}^d$ can be partitioned into two subsets $\mathcal{X}_1$ and $\mathcal{X}_2$ such that the convex hulls of $\mathcal{X}_1$ and $\mathcal{X}_2$ intersect.

Proof: See Mehryar Mohri, Foundations of Machine Learning, Theorem 3.13.

- If two sets are separated by a hyperplane, then their convex hulls are separated.
- So if you have $d+2$ points, Radon says the convex hulls intersect.
- So you cannot shatter the $d+2$ points.
- $d+1$ is okay as we have proved. So the VC dimension is $d+1$.