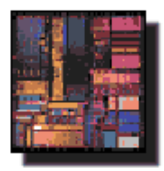
Digital Systems Design Automation

Unit 2: Advanced Boolean Algebra

Lecture 2.2: Boolean Spaces and Functions



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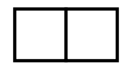
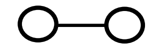
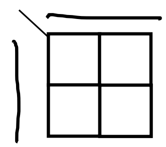
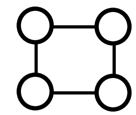
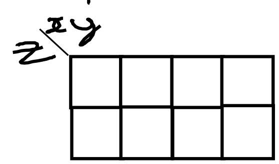
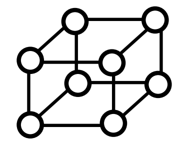
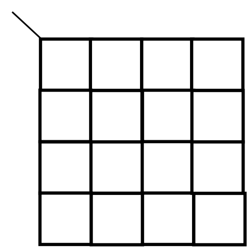
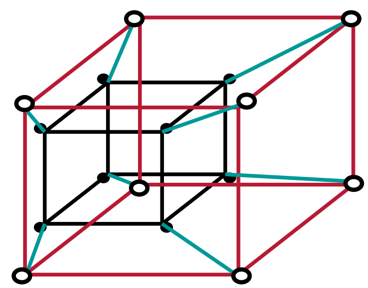
Outline

Boolean Spaces

- Boolean space of n variables is the set of all possible combinations of values that the variables can assume
- Many representations
  - e.g., K-map, n-dimensional unit hypercube

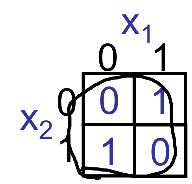
Boolean space	Karnaugh Map	Boolean Hypercube
$B^1 = \{0,1\}$		
$B^2 = B \times B = \{00,01,10,11\}$		
$B^3 = B \times B \times B = \{000,001,010,011,100,101,110,111\}$		
$B^4 = B \times B \times B \times B = \{0000, \ldots 1111\}$		

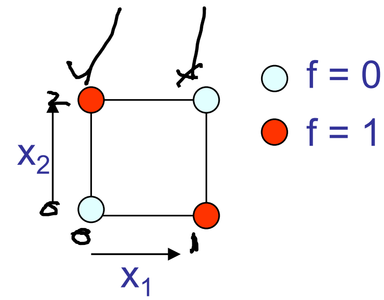
http://en.wikipedia.org/wiki/Karnaugh_map
http://en.wikipedia.org/wiki/Hypercube

Boolean Functions

- Boolean function (a.k.a. logic function) is a mapping from one Boolean space to another
  - E.g., $f(x): B^n \rightarrow B$
  - $x = x_1, x_2, \ldots x_n$ are variables, $x_i \in B$
- On-set of f
  - $\{x \mid f(x) = 1\} = f^1 = f^{-1}(1)$
- Off-set of f
  - $\{x \mid f(x) = 0\} = f^0 = f^{-1}(0)$

Example: $f(x): B^2 \rightarrow B$





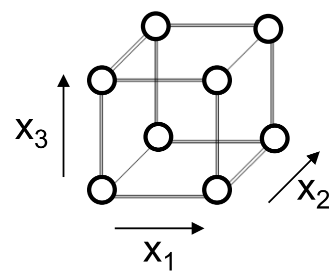
On-set: {01, 10}
Off-set: {00, 11}

Boolean Functions (contd.)

- If $f^1 = B^n$, *i.e.*, $f(x) = 1$, f is a tautology
- If $f^0 = B^n$, *i.e.*, $f(x) = 0$, f is unsatisfiable
- If $f(x) = g(x)$ for all $x \in B^n$, then f and g are equivalent

$B^n \to 2^n$

- Question: How many distinct logic functions of n variables exist?
  - Hint: Think of how many ways you can color the vertices of a Boolean hypercube with two colors



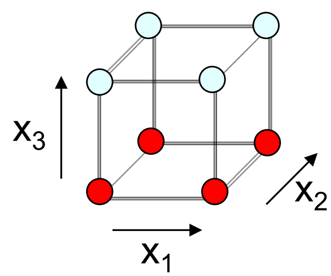
$2^3 = 8$

2^{2^n}

The Set of Boolean Functions

- There are 2^n vertices in input space B^n
  $\rightarrow 2^{2^n}$ distinct logic functions.
  - Assigning each distinct subset of vertices as the on-set ($f^1 \subseteq B^n$) results in a distinct logic function

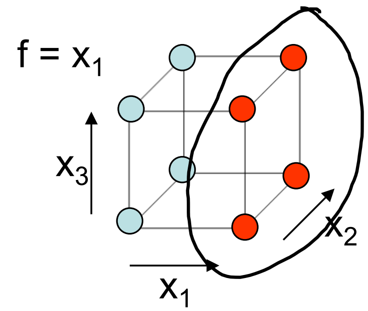
Example:

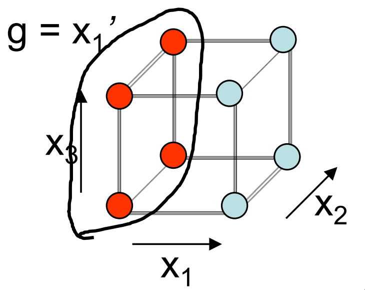


$x_1x_2x_3$	f
0 0 0	1
0 0 1	0
0 1 0	1
0 1 1	0
1 0 0	1
1 0 1	0
1 1 0	1
1 1 1	0

Boolean Functions: A Compositional View

- Another way to think about Boolean functions
  - Compose them using atomic functions and operators
- Atomic functions
  - Constant functions (f = 0, f = 1)
  - Literals
    - A literal is a variable (x_1) or its complement (x_1')
    - Literal x_1 represents the logic function $f = \{x \mid x_1 = 1\}$
    - Literal x_1' represents the logic function $g = \{x \mid x_1 = 0\}$





Notation: $x_1' = \overline{x_1}$

Operations on Boolean Functions

Given two Boolean functions:

$f: B^n \rightarrow B$

$g: B^n \rightarrow B$

Interpretation in terms of on-set and off-set?

- AND operation

  $f \cdot g = \{x \mid f(x)=1 \wedge g(x)=1\}$

- The OR operation

  $f + g = \{x \mid f(x)=1 \vee g(x)=1\}$

- The NOT operation (f')

  $f' = \{x \mid f(x) = 0\}$

The Algebra of Boolean Functions

- The set of all Boolean Functions together with the operations {AND, OR, NOT} also satisfy the laws of Boolean Algebra

Law	Description
Commutativity	$x \lor y = y \lor x$ $x \land y = y \land x$
Associativity	$x \lor (y \lor z) = (x \lor y) \lor z$ $x \land (y \land z) = (x \land y) \land z$
Distributivity	$x \land (y \lor z) = (x \land y) \lor (x \land z)$ $x \lor (y \land z) = (x \lor y) \land (x \lor z)$
Identity	$x \lor 0 = x$ $x \land 1 = x$
Annihilation	$x \land 0 = 0$ $x \lor 1 = 1$
Idempotence	$x \lor x = x$ $x \land x = x$
Absorption	$x \land (x \lor y) = x$ $x \lor (x \land y) = x$

Law	Description
Complementation	$x \land \neg x = 0$ $x \lor \neg x = 1$
Double Negation	$\neg\neg x = x$
De Morgan	$(\neg x) \land (\neg y) = \neg(x \lor y)$ $(\neg x) \lor (\neg y) = \neg(x \land y)$