1 Double-barrier case

Please refer to fig 1 for a schematic.

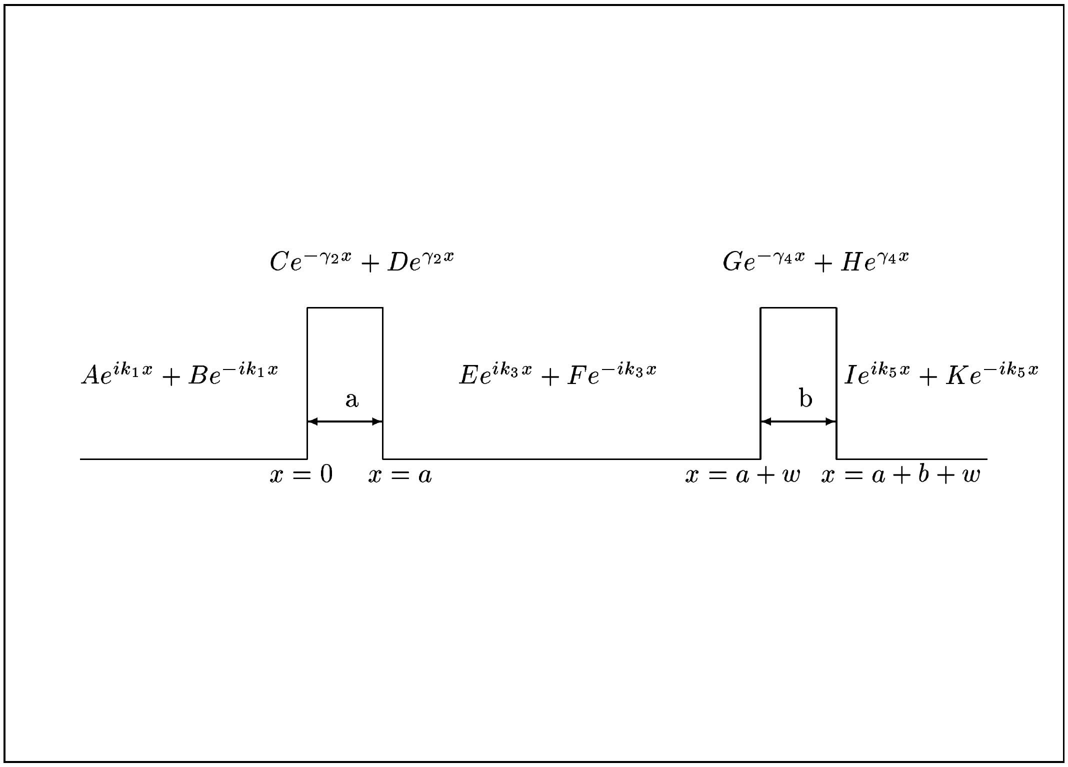

Figure 1: Double barrier case

Matching procedure
$x = 0$ **boundary**
Equating the wave function and the derivative give

$$A + B = C + D \tag{1}$$
$$ik_1(A - B) = -\gamma_2(C - D) \tag{2}$$

Rearrange

$$\begin{bmatrix} A \\ B \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{1}{2}\left(1+i\frac{\gamma_2}{k_1}\right) & \frac{1}{2}\left(1-i\frac{\gamma_2}{k_1}\right) \\ \frac{1}{2}\left(1-i\frac{\gamma_2}{k_1}\right) & \frac{1}{2}\left(1+i\frac{\gamma_2}{k_1}\right) \end{bmatrix}}_{M_1} \begin{bmatrix} C \\ D \end{bmatrix} \tag{3}$$

$x = a$ **boundary**
Equating the wave function and the derivative give

$$\begin{aligned} Ce^{-\gamma_2 a} + De^{\gamma_2 a} &= Ee^{ik_3 a} + Fe^{-ik_3 a} && (4) \\ -\gamma_2\left[Ce^{-\gamma_2 a} - De^{\gamma_2 a}\right] &= ik_3\left[Ee^{ik_3 a} - Fe^{-ik_3 a}\right] && (5) \end{aligned}$$

Rearrange

$$\begin{bmatrix} C \\ D \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{1}{2}\left(1-i\frac{k_3}{\gamma_2}\right)e^{(ik_3+\gamma_2)a} & \frac{1}{2}\left(1+i\frac{k_3}{\gamma_2}\right)e^{-(ik_3-\gamma_2)a} \\ \frac{1}{2}\left(1+i\frac{k_3}{\gamma_2}\right)e^{(ik_3-\gamma_2)a} & \frac{1}{2}\left(1-i\frac{k_3}{\gamma_2}\right)e^{-(ik_3+\gamma_2)a} \end{bmatrix}}_{A} \begin{bmatrix} E \\ F \end{bmatrix} \tag{6}$$

The matrix A can be decomposed into

$$A = \underbrace{\begin{bmatrix} e^{\gamma_2 a} & 0 \\ 0 & e^{-\gamma_2 a} \end{bmatrix}}_{M_2} \underbrace{\begin{bmatrix} \frac{1}{2}\left(1-i\frac{k_3}{\gamma_2}\right) & \frac{1}{2}\left(1-i\frac{k_3}{\gamma_2}\right) \\ \frac{1}{2}\left(1+i\frac{k_3}{\gamma_2}\right) & \frac{1}{2}\left(1-i\frac{k_3}{\gamma_2}\right) \end{bmatrix}}_{M_3} \underbrace{\begin{bmatrix} e^{ik_3 a} & 0 \\ 0 & e^{-ik_3 a} \end{bmatrix}}_{M_4} \tag{7}$$

$x = a + w$ **boundary**
Equating the wave function and the derivative give

$$\begin{aligned} Ee^{ik_3(a+w)} + Fe^{-ik_3(a+w)} &= Ge^{-\gamma_4 a} + He^{\gamma_4 a} && (8) \\ ik_3\left[Ee^{ik_3(a+w)} - Fe^{-ik_3(a+w)}\right] &= -\gamma_4\left[Ge^{-\gamma_4 a} - He^{\gamma_4 a}\right] && (9) \end{aligned}$$

Rearange

$$\begin{aligned} \begin{bmatrix} E \\ F \end{bmatrix} &= \begin{bmatrix} \frac{1}{2}\left(1+i\frac{\gamma_4}{k_3}\right)e^{-(ik_3+\gamma_4)(a+w)} & \frac{1}{2}\left(1-i\frac{\gamma_4}{k_3}\right)e^{(\gamma_4-ik_3)(a+w)} \\ \frac{1}{2}\left(1-i\frac{\gamma_4}{k_3}\right)e^{(ik_3-\gamma_4)(a+w)} & \frac{1}{2}\left(1+i\frac{\gamma_4}{k_3}\right)e^{(ik_3+\gamma_4)(a+w)} \end{bmatrix} \begin{bmatrix} G \\ H \end{bmatrix} \\ &= \underbrace{\begin{bmatrix} e^{-ik_3(a+w)} & 0 \\ 0 & e^{ik_3(a+w)} \end{bmatrix}}_{M_5} \underbrace{\begin{bmatrix} \frac{1}{2}\left(1+i\frac{\gamma_4}{k_3}\right) & \frac{1}{2}\left(1-i\frac{\gamma_4}{k_3}\right) \\ \frac{1}{2}\left(1-i\frac{\gamma_4}{k_3}\right) & \frac{1}{2}\left(1+i\frac{\gamma_4}{k_3}\right) \end{bmatrix}}_{M_6}. \end{aligned}$$

$$
\cdot \quad \underbrace{\begin{bmatrix} e^{-\gamma_4(a+w)} & 0 \\ 0 & e^{\gamma_4(a+w)} \end{bmatrix}}_{M_7} \begin{bmatrix} G \\ H \end{bmatrix} \tag{10}
$$

$x = a + b + w$ **boundary**

$$
Ge^{-\gamma_4(a+b+w)} + He^{\gamma_4(a+b+w)} = Ie^{ik_5(a+b+w)} + Ke^{-ik_5(a+b+w)} \tag{11}
$$

$$
-\gamma_4\Big[Ge^{-\gamma_4(a+b+w)} - He^{\gamma_4(a+b+w)}\Big] = ik_5\Big[Ie^{ik_5(a+b+w)} - Ke^{-ik_5(a+b+w)}\Big] \tag{12}
$$

Let $\alpha = a + b + w$ and rearrange

$$
\begin{bmatrix} G \\ H \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\Big(1 - i\frac{k_5}{\gamma_4}\Big)e^{(ik_5+\gamma_4)\alpha} & \frac{1}{2}\Big(1 + i\frac{k_5}{\gamma_4}\Big)e^{(\gamma_4-ik_5)\alpha} \\ \frac{1}{2}\Big(1 + i\frac{k_5}{\gamma_4}\Big)e^{(ik_5-\gamma_4)\alpha} & \frac{1}{2}\Big(1 - i\frac{k_5}{\gamma_4}\Big)e^{-(ik_5+\gamma_4)\alpha} \end{bmatrix} \begin{bmatrix} I \\ K \end{bmatrix} \tag{13}
$$

$$
\begin{bmatrix} G \\ H \end{bmatrix} = \underbrace{\begin{bmatrix} e^{\gamma_4\alpha} & 0 \\ 0 & e^{-\gamma_4\alpha} \end{bmatrix}}_{M_8} \underbrace{\begin{bmatrix} \frac{1}{2}\Big(1 - i\frac{k_5}{\gamma_4}\Big) & \frac{1}{2}\Big(1 + i\frac{k_5}{\gamma_4}\Big) \\ \frac{1}{2}\Big(1 + i\frac{k_5}{\gamma_4}\Big) & \frac{1}{2}\Big(1 - i\frac{k_5}{\gamma_4}\Big) \end{bmatrix}}_{M_9} \underbrace{\begin{bmatrix} e^{ik_5\alpha} & 0 \\ 0 & e^{-ik_5\alpha} \end{bmatrix}}_{M_{10}} \begin{bmatrix} I \\ k \end{bmatrix} \tag{14}
$$

To summarize, we have

$$
\begin{bmatrix} A \\ B \end{bmatrix} = M_1M_2M_3M_4M_5M_6M_7M_8M_9M_{10} \begin{bmatrix} I \\ K \end{bmatrix} \tag{15}
$$

$$
M_4 \cdot M_5 = \begin{bmatrix} e^{ik_3a} & 0 \\ 0 & e^{-ik_3a} \end{bmatrix} \begin{bmatrix} e^{-ik_3(a+w)} & 0 \\ 0 & e^{ik_3(a+w)} \end{bmatrix} \tag{16}
$$

$$
= \begin{bmatrix} e^{-ik_3w} & 0 \\ 0 & e^{ik_3w} \end{bmatrix} \tag{17}
$$

Similarly

$$
M_7 \cdot M_8 = \begin{bmatrix} e^{\gamma_4 b} & 0 \\ 0 & e^{-\gamma_4 b} \end{bmatrix} \tag{18}
$$

Introduce the following short hand notation

$$
\alpha_{ij} = \frac{1}{2}\Big(1 + i\frac{\gamma_j}{k_i}\Big), \qquad \beta_{ij} = \frac{1}{2}\Big(1 + i\frac{k_j}{\gamma_i}\Big) \tag{19}
$$

Then

$$
\begin{bmatrix} A \\ B \end{bmatrix} = \overbrace{\underbrace{\begin{bmatrix} \alpha_{12} & \alpha_{12}^* \\ \alpha_{12}^* & \alpha_{12} \end{bmatrix}}^{\text{first interface}} \overbrace{\begin{bmatrix} e^{\gamma_2 a} & 0 \\ 0 & e^{-\gamma_2 a} \end{bmatrix}}^{\text{barrier}} \overbrace{\begin{bmatrix} \beta_{23}^* & \beta_{23} \\ \beta_{23} & \beta_{23}^* \end{bmatrix}}^{\text{second interface}}}^{\text{barrier 1}} \overbrace{\begin{bmatrix} e^{-ik_3 w} & 0 \\ 0 & e^{ik_3 w} \end{bmatrix}}^{\text{well}} \cdot
$$

$$
\cdot \underbrace{\underbrace{\begin{bmatrix} \alpha_{34} & \alpha_{34}^* \\ \alpha_{34}^* & \alpha_{34} \end{bmatrix}}_{\text{first interface}} \underbrace{\begin{bmatrix} e^{\gamma_4 b} & 0 \\ 0 & e^{-\gamma_4 b} \end{bmatrix}}_{\text{barrier}} \underbrace{\begin{bmatrix} \beta_{45}^* & \beta_{45} \\ \beta_{45} & \beta_{45}^* \end{bmatrix}}_{\text{second interface}}}_{\text{barrier 2}} \begin{bmatrix} e^{ik_5(a+b+w)} & 0 \\ 0 & e^{-ik_5(a+b+w)} \end{bmatrix} \begin{bmatrix} I \\ K \end{bmatrix} \tag{20}
$$

If we call the first barrire matrix M_L, the well matrix M_W and the second barriwe matrix M_R we get

$$
\begin{bmatrix} A \\ B \end{bmatrix} = M_L M_W M_R \begin{bmatrix} e^{+ik_5\alpha} & 0 \\ 0 & e^{-ik_5\alpha} \end{bmatrix} \begin{bmatrix} I \\ K \end{bmatrix} \tag{21}
$$

$$
= M_T \begin{bmatrix} e^{ik_5\alpha} I \\ e^{-ik_5\alpha} K \end{bmatrix} \tag{22}
$$

Transmission coefficient

Apply the first asymptotic condition, i.e., $K = 0$. Then we get

$$
\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} m_{11}^T & m_{12}^T \\ m_{21}^T & m_{22}^T \end{bmatrix} \begin{bmatrix} e^{ik_5\alpha} I \\ 0 \end{bmatrix} \tag{23}
$$

or

$$
A = m_{11}^T e^{ik_5\alpha} I \tag{24}
$$

or

$$
\boxed{T = \frac{k_5}{k_1} \frac{1}{\left|m_{11}^T\right|^2}} \tag{25}
$$

For each individual barrier we have

$$
M_B = \begin{bmatrix} \alpha & \alpha^* \\ \alpha^* & \alpha \end{bmatrix} \begin{bmatrix} e^{\gamma a} & 0 \\ 0 & e^{-\gamma a} \end{bmatrix} \begin{bmatrix} \beta^* & \beta \\ \beta & \beta^* \end{bmatrix} \tag{26}
$$

As a special case, when $k = k_1 = k_3$ and $\gamma_2 = \gamma$

$$
M_B = \begin{bmatrix} \frac{1}{2}\left(1 + i\frac{\gamma}{k}\right) & \frac{1}{2}\left(1 - i\frac{\gamma}{k}\right) \\ \frac{1}{2}\left(1 - i\frac{\gamma}{k}\right) & \frac{1}{2}\left(1 + i\frac{\gamma}{k}\right) \end{bmatrix} \begin{bmatrix} e^{\gamma a} & 0 \\ 0 & e^{-\gamma a} \end{bmatrix} \begin{bmatrix} \frac{1}{2}\left(1 - i\frac{k}{\gamma}\right) & \frac{1}{2}\left(1 + i\frac{k}{\gamma}\right) \\ \frac{1}{2}\left(1 + i\frac{k}{\gamma}\right) & \frac{1}{2}\left(1 - i\frac{k}{\gamma}\right) \end{bmatrix}
$$

$$= \begin{bmatrix} \frac{1}{2}\left(1+i\frac{\gamma}{k}\right) & \frac{1}{2}\left(1-i\frac{\gamma}{k}\right) \\ \frac{1}{2}\left(1-i\frac{\gamma}{k}\right) & \frac{1}{2}\left(1+i\frac{\gamma}{k}\right) \end{bmatrix} \begin{bmatrix} \frac{1}{2}\left(1-i\frac{k}{\gamma}\right)e^{\gamma a} & \frac{1}{2}\left(1+i\frac{k}{\gamma}\right)e^{\gamma a} \\ \frac{1}{2}\left(1+i\frac{k}{\gamma}\right)e^{-\gamma a} & \frac{1}{2}\left(1-i\frac{k}{\gamma}\right)e^{-\gamma a} \end{bmatrix} \tag{27}$$

if we express

$$M_B = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_2 2 \end{bmatrix} \tag{28}$$

then

$$\begin{aligned} m_{11} &= \frac{1}{2}\left(1+i\frac{\gamma}{k}\right)\frac{1}{2}\left(1-i\frac{k}{\gamma}\right)e^{\gamma a} + \frac{1}{2}\left(1-i\frac{\gamma}{k}\right)\frac{1}{2}\left(1+i\frac{k}{\gamma}\right)e^{-\gamma a} \\ &= \cosh(\gamma a) + \frac{i}{2}\left(\frac{\gamma}{k} - \frac{k}{\gamma}\right)\sinh(\gamma a) \end{aligned} \tag{29}$$

Recall that

$$\cosh\theta = \frac{1}{2}\left(e^{\theta} + e^{-\theta}\right) \tag{30}$$

$$\sinh\theta = \frac{1}{2}\left(e^{\theta} - e^{-\theta}\right) \tag{31}$$

$$\tag{32}$$

$$\begin{aligned} m_{12} &= \frac{1}{2}\left(1+i\frac{\gamma}{k}\right)\frac{1}{2}\left(1+i\frac{k}{\gamma}\right)e^{\gamma a} + \frac{1}{2}\left(1-i\frac{\gamma}{k}\right)\frac{1}{2}\left(1-i\frac{k}{\gamma}\right)e^{-\gamma a} \\ &= \frac{i}{2}\left(\frac{\gamma}{k} + \frac{k}{\gamma}\right)\sinh(\gamma a) \end{aligned} \tag{33}$$

It is straightforward to show that

$$m_{21} = m_{12}^* = -\frac{i}{2}\left(\frac{\gamma}{k} + \frac{k}{\gamma}\right)\sinh(\gamma a) \tag{34}$$

$$m_{22} = m_{11}^* = \cosh(\gamma a) - \frac{i}{2}\left(\frac{\gamma}{k} - \frac{k}{\gamma}\right)\sinh(\gamma a) \tag{35}$$

$$\det(M_B) = m_{11}m_{22} - m_{12}m_{21} \tag{36}$$

$$= |m_{11}|^2 - |m_{12}|^2 \tag{37}$$

$$\vdots \tag{38}$$

$$= \cosh^2(\gamma a) - \sinh^2(\gamma a) \tag{39}$$

$$= 1 \tag{40}$$

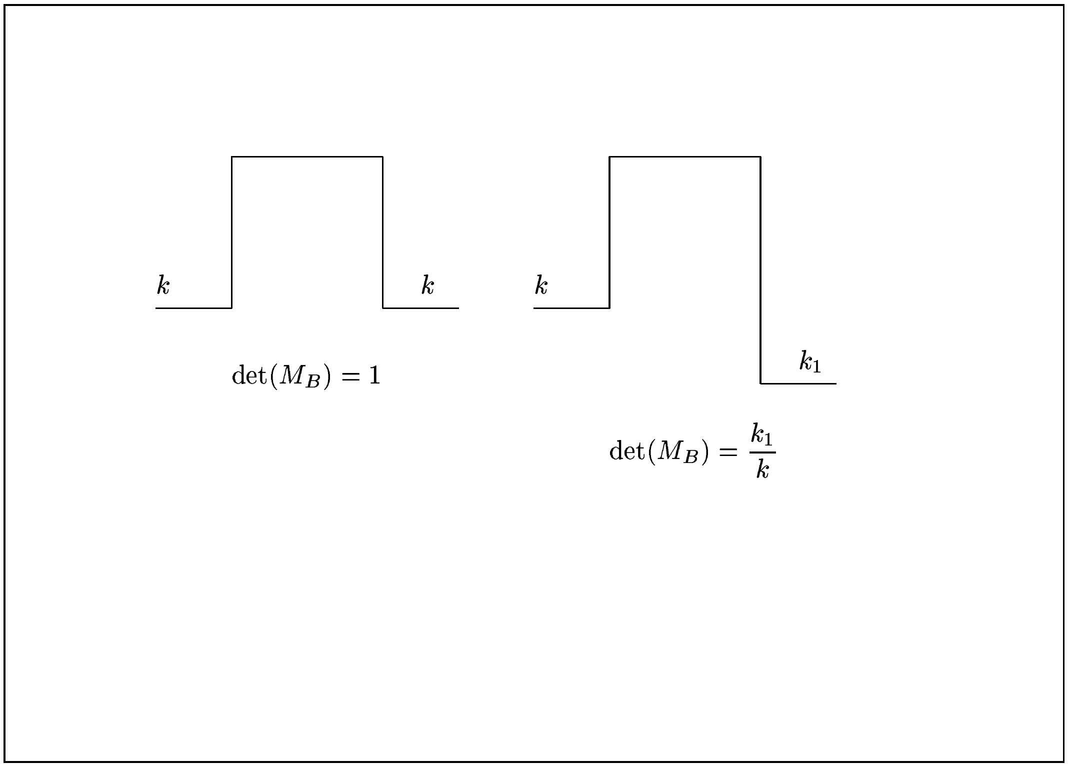

Figure 2: Double barrier case

Now consider the double barrier case. The transmission coefficient is given by

$$T = \frac{k_5}{k_1}\frac{1}{\left|m_{11}^T\right|^2} \tag{41}$$

If $k_5 = k_1 = k_3$ then $T = \frac{1}{\left|m_{11}^T\right|^2}$ where

$$\begin{aligned} M_T &= M_L M_W M_R = \begin{bmatrix} m_{11}^L & m_{12}^L \\ m_{12}^{*L} & m_{11}^{*L} \end{bmatrix} \begin{bmatrix} e^{-ikw} & 0 \\ 0 & e^{ikw} \end{bmatrix} \begin{bmatrix} m_{11}^R & m_{12}^R \\ m_{12}^{*R} & m_{11}^{*R} \end{bmatrix} \\ &= \begin{bmatrix} m_{11}^L & m_{12}^L \\ m_{12}^{*L} & m_{11}^{*L} \end{bmatrix} \begin{bmatrix} m_{11}^R e^{-ikw} & m_{12}^R e^{-ikw} \\ m_{12}^{*R} e^{ikw} & m_{11}^{*R} e^{ikw} \end{bmatrix} \\ &= \begin{bmatrix} m_{11}^L m_{11}^R e^{-ikw} + m_{12}^L m_{12}^{*R} e^{ikw} & m_{12}^T \\ m_{21}^T & m_{22}^T \end{bmatrix} \end{aligned} \tag{42}$$

Therefore

$$\boxed{m_{11}^{T} = m_{11}^{L} m_{11}^{R} e^{-ikw} + m_{12}^{L} m_{12}^{*R} e^{ikw}} \tag{43}$$

Resonance behavior may occurs in this expression through the phase factor which may lead to concellation of terms that minimize m_{11}^T, giving rise to peaked behavior in the transmission coefficient. To see this we will consider two cases.

1. symmetric barriers
2. more general asymmetric case

Symmetric barriers

For the case of symmetric barriers, the propagator constant is the same in the well, left and right regiomns. Also, both barriers have the same width and height. To simplify the notation, we write the matrix elements in polar notation

$$m_{11}^{L} = m_{11}^{R} = |m_{11}| e^{\theta_{11}} \tag{44}$$

where

$$|m_{11}|^2 = \cosh^2(\gamma a) + \frac{1}{4}\left(\frac{\gamma}{k} - \frac{k}{\gamma}\right)^2 \sinh^2(\gamma a) \tag{45}$$

$$\theta_{11} = \tan^{-1}\left[\frac{1}{2}\left(\frac{\gamma}{k} - \frac{k}{\gamma}\right)\tanh(\gamma a)\right] \tag{46}$$

Then the matrix element m_{11}^T equals

$$\begin{aligned}
m_{11}^{T} &= |m_{11}|^2 e^{i2\theta_{11} - ikw} + |m_{12}|^2 e^{ikw} \\
&= e^{i\theta_{11}}\left[|m_{11}|^2 e^{i(\theta_{11} - kw)} + |m_{12}|^2 e^{i(kw - \theta_{11})}\right] \\
&= e^{i\theta_{11}}\left[\left(|m_{11}|^2 + |m_{12}|^2\right)\cos(\theta_{11} - kw)\right. \\
&+ \left. i\underbrace{\left(|m_{11}|^2 - |m_{12}|^2\right)}_{1} \sin(\theta_{11} - kw)\right] \\
&= e^{i\theta_{11}}\left[\left(|m_{11}|^2 + |m_{12}|^2\right)\cos(\theta_{11} - kw) + i\sin(\theta_{11} - kw)\right]
\end{aligned} \tag{47}$$

$$\begin{aligned}
|m_{11}^{T}|^2 &= \left[|m_{11}|^2 + |m_{12}|^2\right]^2 \cos^2(\theta_{11} - kw) + \underbrace{\sin^2(\theta_{11} - kw)}_{1 - \cos^2(\theta_{11} - kw)} \\
&= 1 + \underbrace{\left\{\left[|m_{11}|^2 + |m_{12}|^2\right]^2 - 1\right\}}_{\left(|m_{11}|^2 + |m_{12}|^2 - 1\right)\left(|m_{11}|^2 + |m_{12}|^2 + 1\right)} \cos^2(\theta_{11} - kw)
\end{aligned} \tag{48}$$

We also notice that

$$\begin{aligned}
\det(M_L) &= \det(M_R) = 1 = \left|m_{11}\right|^2 - \left|m_{12}\right|^2 = 1 && (49)\\
\left|m_{11}\right|^2 - 1 &= \left|m_{12}\right|^2 && (50)\\
\left|m_{11}\right|^2 &= 1 + \left|m_{12}\right|^2 && (51)
\end{aligned}$$

Hence

$$\begin{aligned}
\left|m_{11}\right|^2 &= 1 + 2\left|m_{12}\right|^2 \cdot 2\left|m_{11}\right|^2 \cos^2(\theta_{11} - kw)\\
&= 1 + 4\left|m_{11}\right|^2\left|m_{12}\right|^2 \cos^2(\theta_{11} - kw) && (52)
\end{aligned}$$

Consider a single barrier

$$\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{bmatrix} C \\ 0 \end{bmatrix} \qquad (53)$$

$$\begin{aligned}
A &= m_{11} C && (54)\\
B &= m_{21} C = m_{21} \frac{A}{m_{11}} = \frac{m_{21}}{m_{11}} A && (55)
\end{aligned}$$

$$\begin{aligned}
T_1 &= \left|\frac{C}{A}\right|^2 = \frac{1}{\left|m_{11}\right|^2} && (56)\\
R_1 &= \left|\frac{B}{A}\right| = \left|\frac{m_{21}}{m_{11}}\right|^2 \Rightarrow |m_{21}|^2 = R_1 \, |m_{11}|^2 = \frac{R_1}{T_1} && (57)
\end{aligned}$$

Thus

$$\begin{aligned}
T &= \frac{1}{\left|m_{11}^T\right|^2} = \frac{1}{1 + 4\frac{1}{T_1}\frac{R_1}{T_1}\cos^2(\theta_{11} - kw)}\\
&= \frac{T_1^2}{T_1^2 + 4R_1 \cos^2(\theta_{11} - kw)} && (58)
\end{aligned}$$

T_1 and R_1 are the transmission and reflection coefficients of a single symmetric barrier.

minimum $T \Rightarrow$ occurs when $\theta_{11} - kw = n\pi$ which gives

$$\begin{aligned}
T_{\text{min}} &= \frac{T_1^2}{T_1^2 + 4R_1} = \frac{T_1^2}{T_1^2 + 4(1 - T_1)} = \frac{T_1^2}{4 + T_1(T_1 - 1)}\\
&= \frac{T_1^2}{4 - T_1(1 - T_1)} && (59)
\end{aligned}$$

$$T_{\text{min}} \approx \frac{T_1^2}{4} \qquad \text{when } T_1 \text{ is small} \tag{60}$$

Thus the off-resonance transmission appears to be cascaded transmission of two successive idential barriers independent of the well in between.

maximum $T \Rightarrow$ Occurs when $kw - \theta_{11} = (2n+1)\frac{\pi}{2}$ or $\theta_{11} - kw = (2n+1)\frac{\pi}{2}$

$$T_{\text{max}} = \frac{T_1^2}{T_1^2} = 1 \tag{61}$$

We want to find the wave vector, i.e., the energy of the incident particle for which resonant transmission occurs. We will consider for simplicity the case where we have strong barriers, i.e.,

$$\gamma \gg k \tag{62}$$

In this case

$$\begin{aligned} \theta_{11} &= \tan^{-1}\left[\frac{1}{2}\frac{\gamma^2 - k^2}{k\gamma}\tanh(\gamma a)\right] \\ &\to \frac{\pi}{2} \end{aligned} \tag{63}$$

Therefore in this limit

$$\begin{aligned} kw - \theta_{11} &= kw - \frac{\pi}{2} = (2n+1)\frac{\pi}{2} \\ kw &= (2n+1)\frac{\pi}{2} + \frac{\pi}{2} \\ &= (2n+2)\frac{\pi}{2} \\ &= (n+1)\pi \qquad n \text{ an integer} \end{aligned} \tag{64}$$

$kw = m\pi \to$ resonant levels are bound state levels of a finite well formed between the two barriers and pronounced transmission of the electron wave occurs , when the energy of the electron is aligned with one of these quasi bound states.

On resonance, the electron wave is reflected back and forth in a way that adds coherently. The unity transmission due to perfect match between the barriers is broken by any asymmetry.

Asymmetric barriers

For the asymmetric case we have

$$m_{11}^T = m_{11}^L m_{11}^R e^{-ik_3 w} + m_{12}^L m_{21}^R e^{ik_3 w} \tag{65}$$

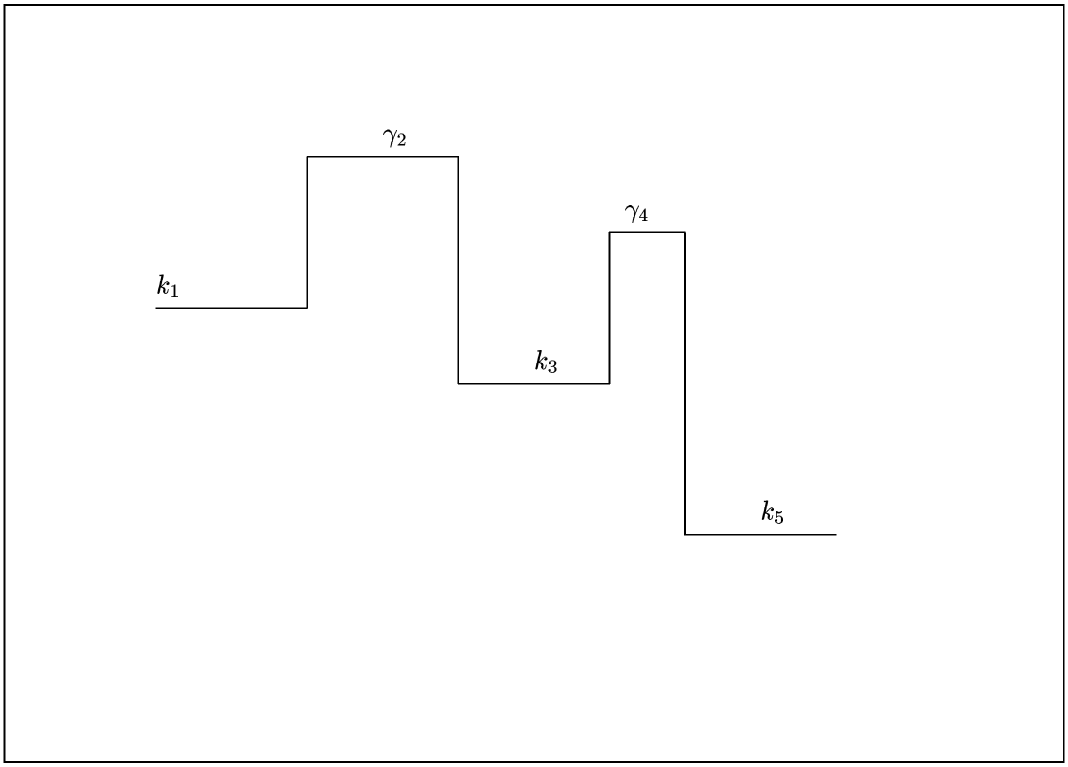

Figure 3: Asymmetric case

Use the polar coordinates

$$m_{ij} = |m_{ij}| \, e^{i\theta_{ij}} \tag{66}$$

to get

$$\begin{aligned} m_{11}^T &= \left|m_{11}^L\right| \left|m_{11}^R\right| e^{i\left(\theta_{11}^L + \theta_{11}^R - k_3 w\right)} \\ &+ \left|m_{12}^L\right| \left|m_{21}^R\right| e^{i\left(\theta_{12} + \theta_{21} - k_3 w\right)} \end{aligned} \tag{67}$$

where $\theta_{12} = \theta_{21} = \pm\frac{\pi}{2}$ (Only true when $k_1 = k_3 = k_5 = k$). Therefore $\theta_{12} + \theta_{21} = \frac{\pi}{2} - \frac{\pi}{2} = 0$

$$\begin{aligned} \left(m_{11}^T\right) &= \left|m_{11}^L\right| \left|m_{11}^R\right| e^{i\left(\theta_{11}^L + \theta_{11}^R - k_3 w\right)} \\ &+ \left|m_{12}^L\right| \left|m_{21}^R\right| e^{ik_3 w} \end{aligned}$$

$$
\begin{aligned}
&= e^{i\left(\frac{\theta_{11}^L+\theta_{11}^R}{2}\right)} \Bigg[\left|m_{11}^L\right| \left|m_{11}^R\right| e^{i\left[\underbrace{\left(\frac{\theta_{11}^L+\theta_{11}^R}{2}\right) - k_3 w}_{\beta}\right]} \\
&+ \left|m_{12}^L\right| \left|m_{21}^R\right| e^{-i\left[\left(\frac{\theta_{11}^L+\theta_{11}^R}{2}\right)+k_3 w\right]}\Bigg] \\
&= e^{i\left(\frac{\theta_{11}^L+\theta_{11}^R}{2}\right)} \left[\left|m_{11}^L\right| \left|m_{11}^R\right| e^{i\beta} + \left|m_{12}^L\right| \left|m_{21}^R\right| e^{-i\beta}\right] \qquad (68)
\end{aligned}
$$

The magnitude of $\left|m_{11}^L\right|$ is

$$
\begin{aligned}
\left|m_{11}^L\right|^2 &= \left[\left|m_{11}^L\right| \left|m_{11}^R\right| + \left|m_{12}^L\right| \left|m_{21}^R\right|\right]^2 \cos^2\beta \\
&+ \left[\left|m_{11}^L\right| \left|m_{11}^R\right| - \left|m_{12}^L\right| \left|m_{21}^R\right|\right]^2 \sin^2\beta \\
&= \left|m_{11}^L\right|^2 \left|m_{11}^R\right|^2 \left\{\left[1 + \frac{\left|m_{12}^L\right| \left|m_{21}^R\right|}{\left|m_{11}^L\right| \left|m_{11}^R\right|}\right]^2 \cos^2\beta \right. \\
&+ \left. \left[1 - \frac{\left|m_{12}^L\right| \left|m_{21}^R\right|}{\left|m_{11}^L\right| \left|m_{11}^R\right|}\right]^2 \sin^2\beta \right\} \qquad (69)
\end{aligned}
$$

$$
\begin{aligned}
(1+\alpha)^2\cos^2\beta &+ (1-\alpha)^2\sin^2\beta \\
&= (1+\alpha)^2\cos^2\beta + (1-\alpha)^2\left[1-\cos^2\beta\right] \\
&= (1-\alpha)^2\left[(1+\alpha)^2 - (1-\alpha)^2\right]\cos^2\beta \\
&= (1-\alpha)^2 + (1+\alpha+1-\alpha)(1+\alpha-1+\alpha)\cos^2\beta \\
&= (1-\alpha)^2 + 2\cdot 2\cos^2\beta \\
&= (1-\alpha)^2 + 4\gamma\cos^2\beta \qquad (70)
\end{aligned}
$$

Therefore

$$
\begin{aligned}
T &= \frac{k_5}{k_1}\frac{1}{\left|m_{11}^T\right|^2} \\
&= \frac{k_5}{k_1} \div \left[\left|m_{11}^L\right|^2 \left|m_{11}^R\right|^2 \left\{\left(1 - \frac{\left|m_{12}^L\right| \left|m_{12}^R\right|}{\left|m_{11}^L\right| \left|m_{11}^R\right|}\right)\right.\right. \\
&+ \left.\left. \frac{\left|m_{12}^L\right| \left|m_{12}^R\right|}{\left|m_{11}^L\right| \left|m_{11}^R\right|}\cos^2\beta\right\}\right]
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{k_5}{k_1} \div \left[\frac{1}{T_1} \frac{1}{T_2} \left\{ \left(1 - \sqrt{\frac{R_1}{T_1}} \sqrt{\frac{R_2}{T_2}} \sqrt{T_1} \sqrt{T_2} \right)^2 \right. \right. \\
&+ \left. \left. 4 \sqrt{\frac{R_1}{T_1}} \sqrt{\frac{R_2}{T_2}} \sqrt{T_1} \sqrt{T_2} \cos^2 \beta \right\} \right] \\
&= \frac{k_5}{k_1} \frac{T_1 T_2}{\left(1 - \sqrt{R_1 R_2}\right)^2 + 4\sqrt{R_1 R_2} \cos^2 \beta}, \qquad \beta = \frac{1}{2} \left(\theta_{11}^L + \theta_{11}^R\right) - k_3 w
\end{aligned}
\tag{71}
$$

Maximum $\rightarrow \beta = (2n+1) \cdot \frac{\pi}{2}$
Maximum transmission

$$
\begin{aligned}
T_{\max} &= \frac{T_1 T_2}{\left(1 - \sqrt{R_1 R_2}\right)^2} \\
&= \frac{T_1 T_2}{\left[1 - \sqrt{(1-T_1)(1-T_2)}\right]^2} \\
&\cong \frac{T_1 T_2}{\left[1 - \left(1 - \frac{T_1}{2}\right)\left(1 - \frac{T_2}{2}\right)\right]^2} \\
&\cong \frac{T_1 T_2}{\left[1 - 1 + \frac{T_1}{2} + \frac{T_2}{2}\right]^2} \\
&\cong \frac{4 T_1 T_2}{\left(T_1 + T_2\right)^2}
\end{aligned}
\tag{72}
$$

$$
T_{\max} \cong \frac{4T_{(\min)}}{T_{(\max)}}
\tag{73}
$$

Minimum $\rightarrow \beta = n\pi$

$$
\begin{aligned}
T_{\min} &= \frac{T_1 T_2}{\left(1 - \sqrt{R_1 R_2}\right)^2 + 4\sqrt{R_1 R_2}} \\
&= \frac{T_1 T_2}{\left(1 + \sqrt{R_1 R_2}\right)^2} \\
&= \frac{T_1 T_2}{\left[1 + \sqrt{(1-T_1)(1-T_2)}\right]^2} \\
&= \frac{T_1 T_2}{\left[1 + \left(1 - \frac{T_1}{2}\right)\left(1 - \frac{T_2}{2}\right)\right]^2} \\
&\cong \frac{T_1 T_2}{4}
\end{aligned}
\tag{74}
$$

The last result indicates that for energies off-resonance, the well does not play an essential role, i.e., the double -barrier structure behaves as two independent barriers.

Summary

Off resonance: → wave function decays exponentially in the classically forbidden region from the incident to the transmitted region, which is consistent with the product of two transmission coefficients.

On resonance: → solutions in the barrier that exponentially grow from the left and the right regions, which allows transmission to be large.

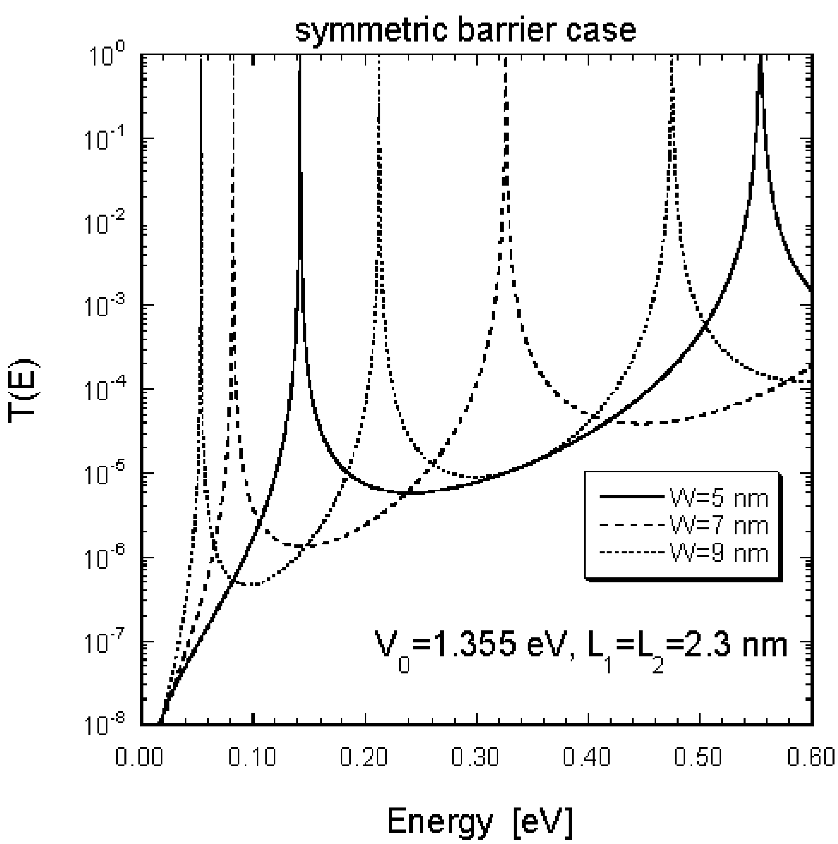

Figure 4: Tunneling probability of a symmetric double-barrier structure. Notice that when the energy of the particle coincides with the energy level in the well, the transmission coefficient equals unity.

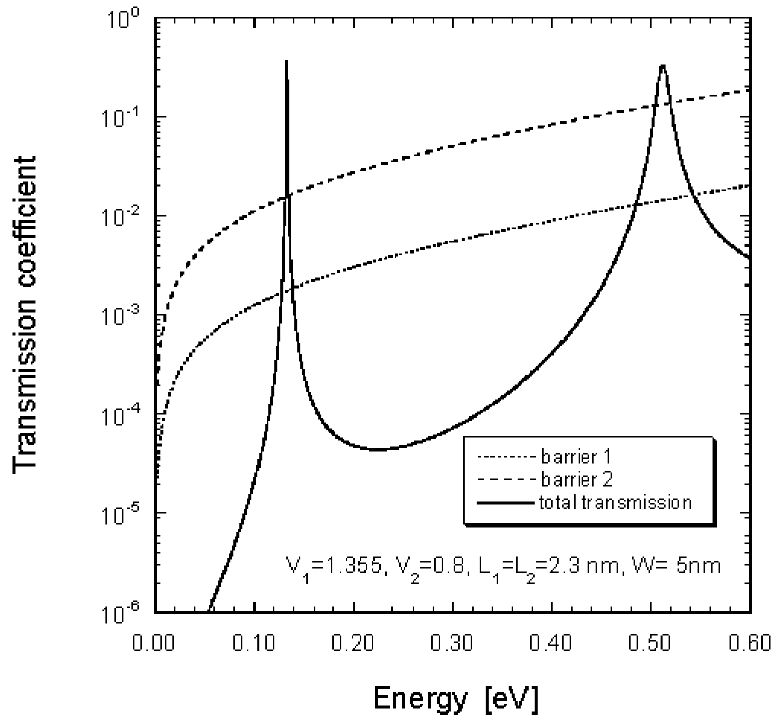

Figure 5: Tunneling probability of a non-symmetric double-barrier structure. Notice that the transmission coefficient is always smaller than 1

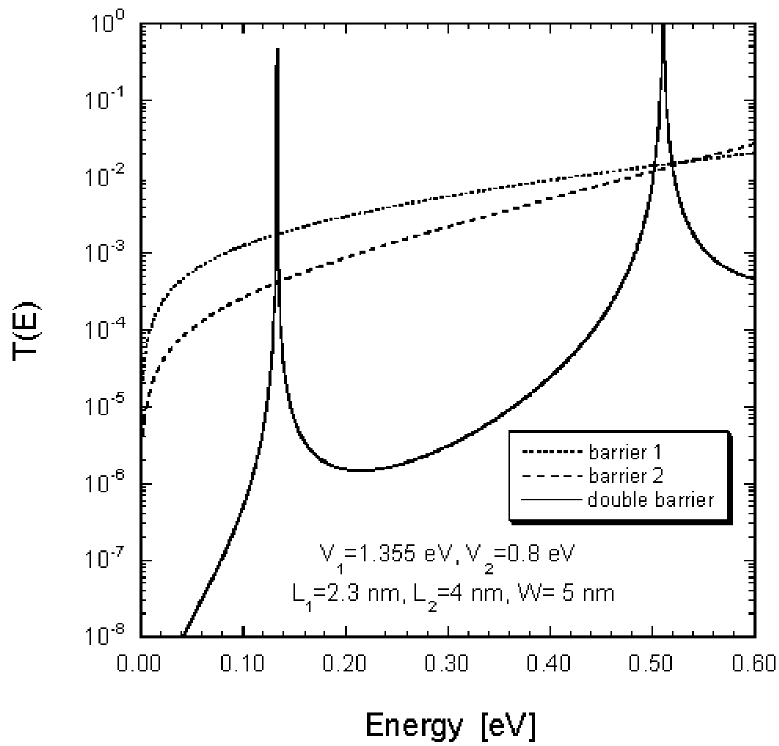

Figure 6: Tunneling probability of a non-symmetric double-barrier structure. Notice that the transmission coefficient is always smaller than 1, but when the energy of the particle coincides with the energy level in the well, it approaches unity when the individual transmission coefficients of the barriers are close to each other.

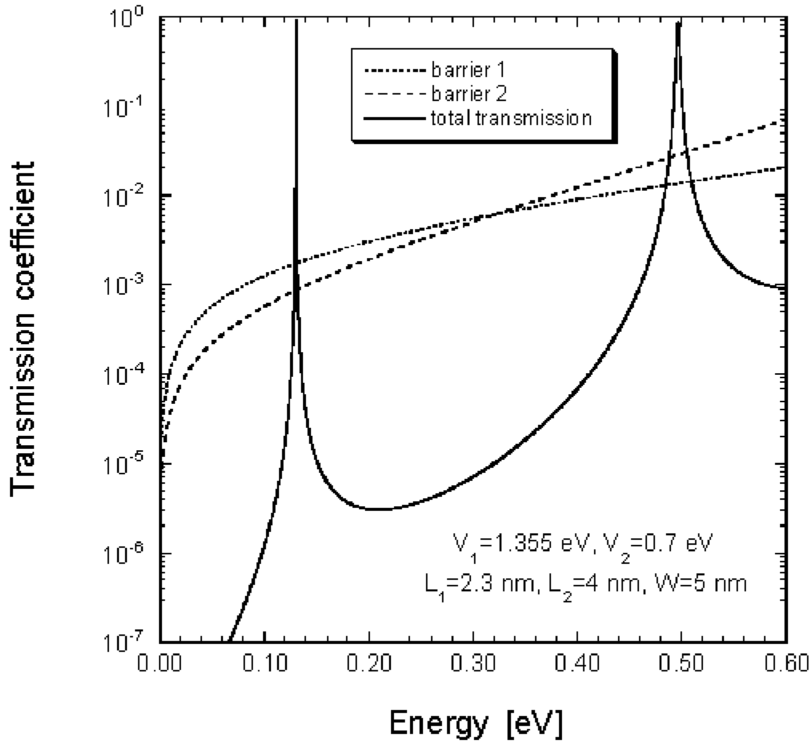

Figure 7: Tunneling probability of a non-symmetric double-barrier structure. Notice that at resonance the total transmission coefficient approaches unity, since the transmission coefficients of each of the barriers are close to each other (they behave as almost identical barriers).

Application - Resonant tunneling diode

• Coherent tunneling → energy conserving process. The properties of the barrier are completely specified by the transmission coefficient.

• Additional constraint in the analysis → transverse momentum is conserved before and after the tunneling process.

• Graphical description of the tunneling process for the case when $k_x = k_y = 0$

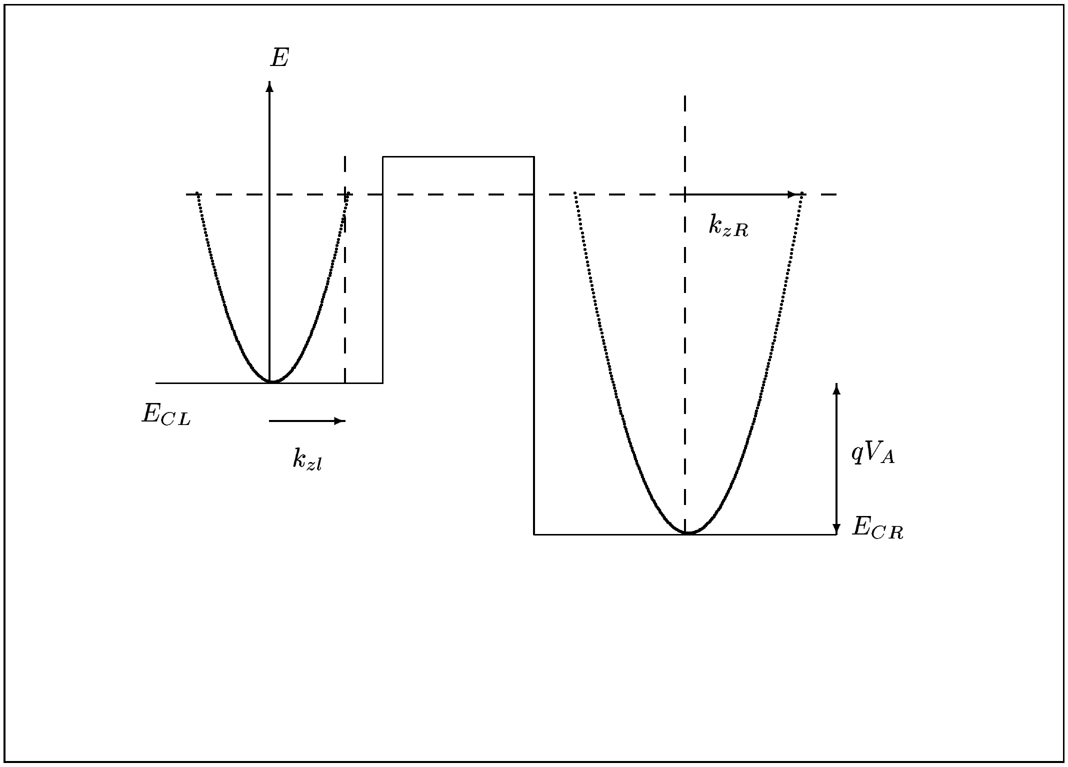

Figure 8: Graphical description of tunneling process

• Description in terms of Fermi levels.

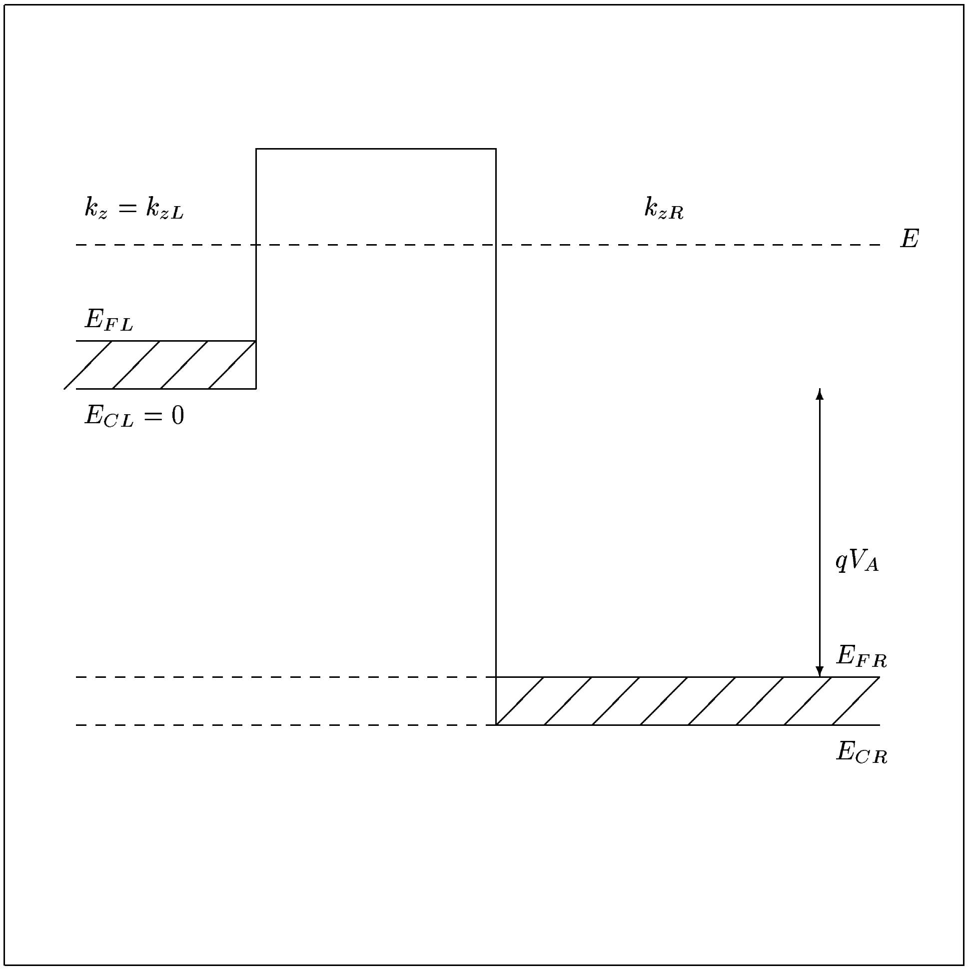

Figure 9: Graphical description of tunneling process in terms of Fermi levels. $\boxed{E_F R = E_{FL} - qV_A}$

- From the energy conservatoin requirement, we have

$$\begin{aligned} E = E_{zL} + E_{tL} &= E_{zR} + E_{tR} + E_{CR} \\ &= E_{zR} + E_t - qV_A \end{aligned} \quad (75)$$

Use the conservation of the transverse momentum

$$E_{tR} = E_{tR} = E_t \quad (76)$$

to get

$$E = E_{zL} + E_t = E_{zR} + E_t - qV_A \tag{77}$$

$$E_{zL} = E_{zR} - qV_A \Rightarrow \boxed{\frac{\hbar^2 k_{zL}^2}{2m} = \frac{\hbar^2 k_{zR}^2}{2m} - qV_A} \tag{78}$$

• To connect the quantum mechanical fluxes to charge current, we need to introduce the statistical mechanical distribution function that describes the occupancy of the current-carrying states. For simplicity we will assure that these distributions are given by the equilibrium Fermi-Dirac function determined by the bulck Fermi levels on the respective sides of the barrier

$$f_{L,R}(E_Z, E_t) = \frac{1}{1 + \exp\left(\dfrac{E_Z + E_t - E_{FL,R}}{k_B T_L}\right)} \tag{79}$$

• As a further approximation, one needs to introduce irreversibility into the formalism. For this case, contacts are assumed to be perfectly absorbing and a particle injected from one side reaches the contact region of the other side, its phase coherence and excess energy are lost through inelastic collisions with the Fermi sea of electrons in the contacts.

⇒ In this picture, current flow is essentially the net difference between the number of particles per unit time transmitted to the right and collected with those transmitted to the left.

• The expression for the current we are familiar with is

$$J \sim nev_z \tag{80}$$

where n is the electron density and v_z is the carrier velocity. This form assures that carriers travel with equal velocity. To get more accurate expression for the current, we start with

$$\begin{aligned} n &= \int g(E) f(E) dE \\ &= 2 \int \frac{d^3k}{(2\pi)^3} f(E_k) \end{aligned} \tag{81}$$

Therefore a more general expression for the current is

$$J = -2e \int \frac{d^3k}{(2\pi)^3} v_z f(E_k) \tag{82}$$

In the presence of tunneling, we need to take into account the tunneling and the forward and backward propagatiomn of carriers. Therefore

$$
\begin{aligned}
J_{L\rightarrow R} &= -\frac{2e}{(2\pi)^3}\int d^3k v_{z,L} T(E_{zL}, E_t) f(E, E_{F_L}) && (83)\\
J_{R\rightarrow L} &= -\frac{2e}{(2\pi)^3}\int d^3k v_{z,R} T(E_{zR}, E_t) f(E, E_{F_R}) && (84)
\end{aligned}
$$

The net current is thus given by

$$
\begin{aligned}
J &= J_{L\rightarrow R} - J_{R\rightarrow L}\\
&= -\frac{2e}{(2\pi)^3}\int d^3k T(E_z)\left[v_{zL} f(E_{F_L}) - v_{zR} f(E_{F_R})\right] && (85)
\end{aligned}
$$

Use the result for the transverse momentum

$$
k_{z_L} dk_{z_L} = k_{z_R} dk_{z_R} \qquad mv = \hbar k \Rightarrow v = \frac{\hbar k}{m} \tag{86}
$$

$$
v_{zL} dk_{zL} = v_{zR} dk_{zR} = v_z dk_z = \frac{\hbar k_z}{m} k_z \tag{87}
$$

Parabolic bands:

$$
E_z = \frac{\hbar^2 k_z^2}{2m} \Rightarrow dE_z = \frac{\hbar^2}{m} k_z dk_z \Rightarrow k_z dk_z = \frac{m}{\hbar^2} dE_z \tag{88}
$$

Also

$$
f(E_{FL}) = \frac{1}{1+\exp\left(\dfrac{E_L - E_{FL}}{k_B T}\right)} \tag{89}
$$

$$
\begin{aligned}
f(E_{FR}) &= \frac{1}{1+\exp\left(\dfrac{E_R - E_{FR}}{k_B T}\right)}\\
&= \frac{1}{1+\exp\left(\dfrac{E_L - E_{FL} + qV_A}{k_B T}\right)} && (90)
\end{aligned}
$$

If we use all of the above results, we arrive at the following expression for the net current

$$
\begin{aligned}
J &= -\frac{2e}{(2\pi)^3}\int_{-\infty}^{+\infty} dk_z \int_0^{\infty} k_t dk_t \int_0^{2\pi} d\phi T(E_z) v_z \left[f(E_{FL}) - f(E_{FR})\right]\\
&= -\frac{2e}{(2\pi)^3} 2\pi \int_{-\infty}^{+\infty} T(E_z) v_z dk_z \int_0^{\infty} k_t dk_t \times
\end{aligned}
$$

$$\times \quad \left[\frac{1}{1+\exp\left(\frac{E_z+E_t-E_F}{k_BT}\right)} - \frac{1}{1+\exp\left(\frac{E_z+E_t-E_F+qV_Z}{k_BT}\right)} \right] \tag{91}$$

$$k_t dk_t = \frac{m}{\hbar^2} dE_t \tag{92}$$

$$\begin{aligned} J &= -\frac{e}{2\pi^2} \int_0^\infty \frac{1}{\hbar} dE_z T(E_z) \int_0^\infty \frac{m}{\hbar^2} dE_t \times \\ &\times \left[\frac{1}{1+\exp\left(\frac{E_z+E_t-E_F}{k_BT}\right)} - \frac{1}{1+\exp\left(\frac{E_z+E_t-E_F+qV_Z}{k_BT}\right)} \right] \end{aligned} \tag{93}$$

The integrals of the form

$$\begin{aligned} \int_0^\infty \frac{dE}{1+\exp\left(\frac{E+\alpha}{k_BT}\right)} &= \int_0^\infty \frac{\exp\left(-\frac{E+\alpha}{k_BT}\right)(-k_BT)d\left(-\frac{E}{k_BT}\right)}{1+\exp\left(-\frac{E+\alpha}{k_BT}\right)} \\ &= -k_BT \ln\left[1+\exp\left(-\frac{E+\alpha}{k_BT}\right)\right]\Big|_0^\infty \\ &= -k_BT\left\{\ln 1 - \ln\left[1+\exp\left(-\frac{\alpha}{k_BT}\right)\right]\right\} \\ &= k_BT \ln\left[1+\exp\left(-\frac{\alpha}{k_BT}\right)\right] \end{aligned} \tag{94}$$

Using this result we arrive at

$$\begin{aligned} J &= -\frac{emk_BT}{2\pi^2\hbar^3} \int_0^\infty dE_z T(E_z) \times \\ &\times \left\{ \ln\left[1+\exp\left(-\frac{E_z-E_F}{k_BT}\right)\right] - \ln\left[1+\exp\left(-\frac{E_z-E_F+qV_A}{k_BT}\right)\right] \right\} \\ &= -\frac{em^*k_BT}{2\pi^2\hbar^3} \int_0^\infty dE_z T(E_z) \ln\left\{ \frac{1+\exp\left(\frac{E_F-E_z}{k_BT}\right)}{1+\exp\left(\frac{E_F-E_z-qV_A}{k_BT}\right)} \right\} \end{aligned} \tag{95}$$

The last expression is known as *Tsu - Esaki* formula, where the particular form was popularized in connection to resonant tunneling devices.

Logarithmic term $\Rightarrow$ $\boxed{\text{Supply function}}$ $\rightarrow$ determines the relative wight of available carriers at a given perpendicular energy.

Low - T limit:

$$\begin{aligned}\ln\left\{1 \right. &+ \left.\exp\left(\frac{E_F - E_z}{k_B T}\right)\right\} \\ &\rightarrow \frac{1}{k_B T}(E_F - E_z)\theta(E_F - E_z), \qquad E_z < E_F \end{aligned} \tag{96}$$

$$\begin{aligned}\ln\left\{1 \right. &+ \left.\exp\left(\frac{E_F - E_z - qV_A}{k_B T}\right)\right\} \\ &\rightarrow \frac{1}{k_B T}(E_F - E_z - qV_A)\theta(E_F - E_z - qV_A), \qquad E_z < E_F - qV_A \end{aligned} \tag{97}$$

Therefore

$$\begin{aligned} J &= -\frac{em^* k_B T}{2\pi^2\hbar^3}\cdot\frac{1}{k_B T}\left\{\int_0^{E_F} dE_z T(E_z)\cdot(E_F - E_z)\right. \\ &- \left.\int_0^{E_F - qV_A} dE_z T(E_z)\cdot(E_F - E_z - qV_A)\right\} \end{aligned} \tag{98}$$

- Application to resonant tunneling diodes

$$T(E_z) = \delta(E_z - E_0) \tag{99}$$

Under bias conditions, assuming that half of the voltage drop is accross each of the barriers, we have

$$E_0 \rightarrow E_0 - q\frac{V_A}{2} \tag{100}$$

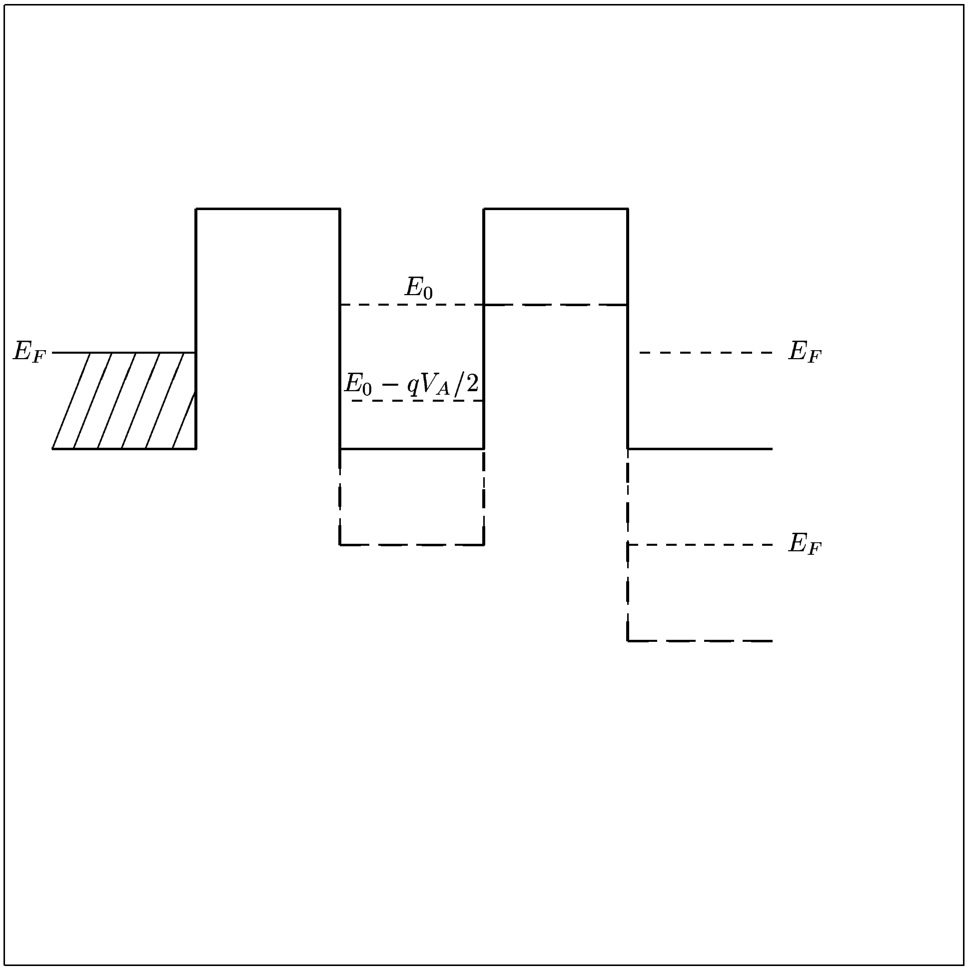

Figure 10: Resonant diode with and without applied voltage

$$T(E_z) = \delta\left(E_z - E_0 + q\frac{V_A}{2}\right) \tag{101}$$

For resonant tunneling diodes at $T = 0K$ the current expression simplifies to

$$J \quad = \quad -\frac{em^*}{2\pi^2\hbar^3} \cdot \left\{\int_0^{E_F} dE_z \delta\left(E_z - E_0 + q\frac{V_A}{2}\right) \cdot (E_F - E_z)\right.$$

$$- \int_0^{E_F - qV_A} dE_z \delta\left(E_z - E_0 + q\frac{V_A}{2}\right) \cdot (E_F - E_z - qV_A)\Bigg\} \quad (102)$$

$$\int_0^{E_F} dE_z \delta\left(E_z - E_0 + q\frac{V_A}{2}\right) \cdot (E_F - E_z) \text{ non zero when} \quad (103)$$

$$\begin{aligned} E_z &= E_0 - q\frac{V_A}{2} \text{ or} \\ 0 &< E_0 - q\frac{V_A}{2} < E_F \end{aligned} \quad (104)$$

$$\begin{aligned} (1) \qquad & E_0 - q\frac{V_A}{2} > 0 \\ & E_0 > q\frac{V_A}{2} \Rightarrow \boxed{V_A < \frac{2}{q}E_0} \end{aligned} \quad (105)$$

$$\begin{aligned} (2) \qquad & E_0 - q\frac{V_A}{2} < E_F \\ & E_0 - E_F < q\frac{V_A}{2} \\ & \boxed{V_A > \frac{2}{q}(E_0 - E_F)} \end{aligned} \quad (106)$$

$$\int_0^{E_F - qV_A} dE_z \delta\left(E_z - E_0 + q\frac{V_A}{2}\right) \cdot (E_F - E_z - qV_A) \text{ non zero when} \quad (107)$$

$$\begin{aligned} E_z &= E_0 - q\frac{V_A}{2} \text{ or} \\ 0 &< E_0 - q\frac{V_A}{2} < E_F - qV_A \end{aligned} \quad (108)$$

$$\begin{aligned} (1) \qquad & E_0 - q\frac{V_A}{2} > 0 \\ & \boxed{V_A < \frac{2}{q}E_0} \end{aligned} \quad (109)$$

$$\begin{aligned} (2) \qquad & E_0 - q\frac{V_A}{2} < E_F - qV_A \\ & \text{use } V_A \neq 0 \Rightarrow E_0 < E_F \end{aligned} \quad (110)$$

This term will contribute when $E_0 < E_F$, which is not usually the case and we can ignore it.

Hence considering only the contributions from the first term, we get

$$J = -\frac{em^*}{2\pi^2\hbar^3}\cdot\left(E_F - E_0 + \frac{eV_A}{2}\right) \qquad \text{for} \qquad \frac{2}{q}(E_0 - E_F) < V_A < \frac{2E_0}{q} \quad (111)$$

1.

$$\begin{aligned} V_A &= \frac{2}{q}(E_0 - E_F) \\ &\Rightarrow \; E_F - E_0 + \frac{q}{2}\frac{2}{q}(E_0 - E_F) = 0, \qquad J_{\text{min}} = 0 \end{aligned} \quad (112)$$

2.

$$\begin{aligned} V_A &= \frac{2E_0}{q} \\ &\Rightarrow \; E_F - E_0 + \frac{q}{2}\frac{2}{q}E_0 = E_F, \qquad J_{\text{max}} = -\frac{em^*}{2\pi^2\hbar^3}E_F \end{aligned} \quad (113)$$

The plot of the current for different temperatures and different Fermi energies is shown in the figure below.

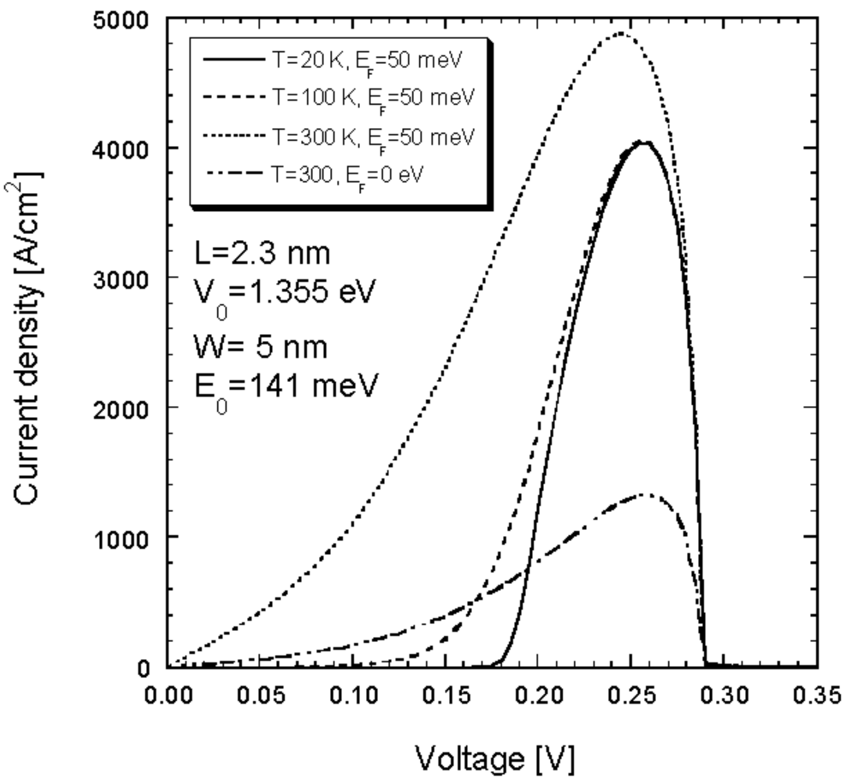

Figure 11: Tunneling current in a resonant tunneling diode obtained by using the Tsu-Esaki simplified model and assuming coherent tunneling. In this example the Fermi-level has been fixed. In a more realistic situation, the Fermi level needs to be calculated based on the doping in the source and detector regions. Notice that at T=300 K, when the Fermi level moves down, the current decreases, which is what is going on in a realistic structure. Increasing the temperature leads to a shift of the Fermi level downwards. This in turn, leads to a reduction of the peak current when compared to the low-temperature situation.