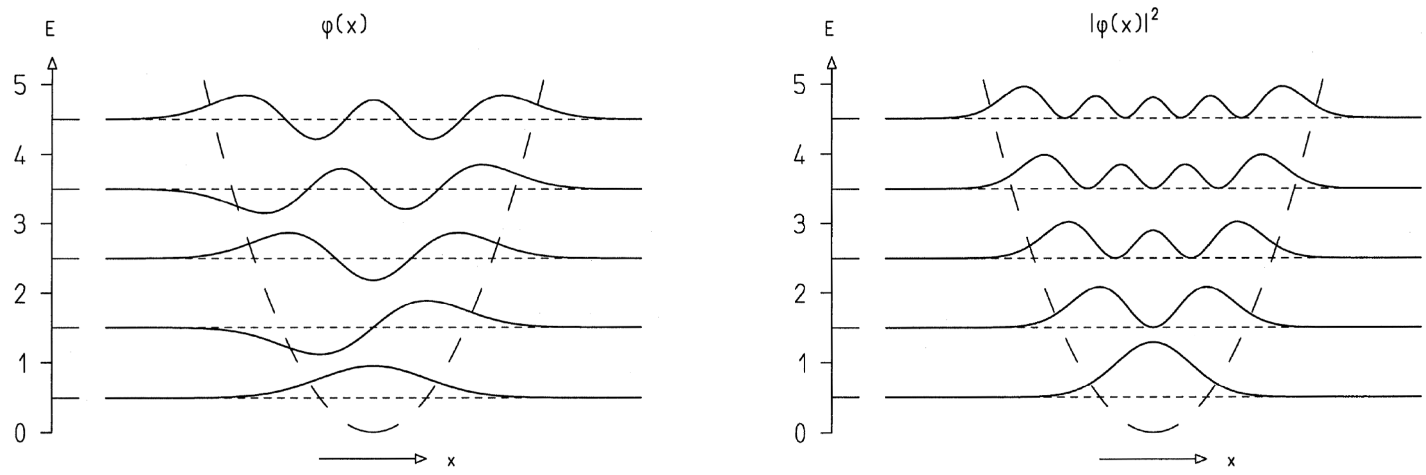
# The Operator Approach

# The Operator Approach

Previously we solved the Schrödinger equation to obtain the wavefunctions and energy levels of a quantum-mechanical HARMONIC OSCILLATOR

* The energy levels are QUANTIZED in terms of the oscillator frequency *w* while the wavefunctions correspond to HERMITE POLYNOMIALS with a FINITE range

$$E_n = \left[ n + \frac{1}{2} \right] \hbar\omega , \quad n = 0,1,2,\ldots \qquad (13.27)$$




***ENERGY LEVELS, WAVEFUNCTIONS* (LEFT) AND *PROBABILITY DENSITY* (RIGHT) FOR THE QUANTUM HARMONIC OSCILLATOR**
***THE PICTURE BOOK OF QUANTUM MECHANICS, S. BRANDT and H-D. DAHMEN, SPRINGER-VERLAG, NEW YORK (1995)***

# The Operator Approach

Today we want to take an ALTERNATIVE approach to this problem that will lead us to the SAME conclusions!

* We start from the time-independent Schrödinger equation for the harmonic oscillator

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi(x)}{\partial x^2}+\frac{1}{2}m\omega^2x^2\psi(x)=E\psi(x) \qquad (13.6)$$

* This can be rewritten in a somewhat more SUGGESTIVE form

$$\frac{1}{2m}\left[\left[\frac{\hbar}{i}\frac{\partial}{\partial x}\right]^2+(m\omega x)^2\right]\psi(x)=E\psi(x) \qquad (14.1)$$

* The idea is to FACTOR the term on the LHS of Equation 14.1 and to this extent we remember that for NUMBERS we can write

$$u^2+v^2=(u-iv)(u+iv) \qquad (14.2)$$

# The Operator Approach

Motivated by Equation 14.2 we define the new OPERATORS

$$a_{\pm} = \frac{1}{\sqrt{2m}}\left[\frac{\hbar}{i}\frac{\partial}{\partial x} \pm im\omega x\right] \qquad (14.3)$$

* It is easy[†] to show that MULTIPLICATION of these two operators yields the following result

$$a_{-}a_{+} = \frac{1}{2m}\left[\left[\frac{\hbar}{i}\frac{\partial}{\partial x}\right]^2 + (m\omega x)^2\right] + \frac{1}{2}\hbar\omega \qquad (14.4)$$

* If we REVERSE the order of multiplication however we obtain the following result

$$a_{+}a_{-} = \frac{1}{2m}\left[\left[\frac{\hbar}{i}\frac{\partial}{\partial x}\right]^2 + (m\omega x)^2\right] - \frac{1}{2}\hbar\omega \qquad (14.5)$$

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†: ***KEEP THE ORDER OF THE OPERATORS AND NOTE THAT d(fx)dx = x(dfldx) + f***

# The Operator Approach

Comparison of Equations 14.4 & 14.5 reveals that the result of multiplying the operators depends on the ORDER in which they are multiplied

$$a_- a_+ - a_+ a_- = \hbar\omega \qquad (14.6)$$

* With these definitions our original Schrödinger equation (Equation 14.1) can now be REWRITTEN as

$$\left[a_+ a_- + \frac{1}{2}\hbar\omega\right]\psi(x) = E\psi(x) \qquad (14.7)$$

* We now notice something interesting if we perform the following operation

$$\left[a_+ a_- + \frac{1}{2}\hbar\omega\right]a_+\psi(x) \qquad (14.8)$$

# The Operator Approach

We can EXPAND Equation 14.8 so that it yields

$$\left[a_+a_- + \frac{1}{2}\hbar\omega\right]a_+\psi(x) = \left[a_+a_-a_+ + \frac{1}{2}\hbar\omega a_+\right]\psi(x) = a_+\left[a_-a_+ + \frac{1}{2}\hbar\omega\right]\psi(x)$$

$$= a_+\left[\left[a_-a_+ - \frac{1}{2}\hbar\omega\right]\psi(x) + \hbar\omega\psi(x)\right] = a_+\left[E\,\psi(x) + \hbar\omega\psi(x)\right]$$

$$= (E + \hbar\omega)\,a_+\psi(x) \qquad (14.9)$$

* Equation 14.9 shows that if $\psi(x)$ satisfies the Schrödinger equation with energy $E$ then $a_+\psi(x)$ ALSO satisfies the Schrödinger equation but with energy $E + h\omega$

* If we know one initial solution to the Schrödinger equation then we can use the operator $a_+$ to determine ALL OTHER solutions!

# The Operator Approach

ALTERNATIVELY we could use the operator $a_-$ to demonstrate that

$$\left[a_-a_+ - \frac{1}{2}\hbar\omega\right]a_-\psi(x) = \left[a_-a_+a_- - \frac{1}{2}\hbar\omega a_-\right]\psi(x) = a_-\left[a_+a_- - \frac{1}{2}\hbar\omega\right]\psi(x)$$

$$= a_-\left[\left[a_+a_- + \frac{1}{2}\hbar\omega\right]\psi(x) - \hbar\omega\psi(x)\right] = a_-\left[E\,\psi(x) - \hbar\omega\psi(x)\right]$$

$$= (E - \hbar\omega)\,a_-\psi(x) \qquad (14.10)$$

* Equation 14.10 shows that if $\psi(x)$ satisfies the Schrödinger equation with energy $E$ then $a_-\psi(x)$ ALSO satisfies the Schrödinger equation but with energy $E$ - h$\omega$

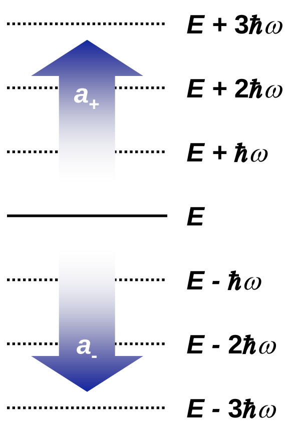
* If we know one initial solution to the Schrödinger equation then we can equally use the operator $a_-$ to determine ALL OTHER solutions

# The Operator Approach

The operators $a_{\pm}$ are referred to as LADDER OPERATORS since they allow us to determine ALL allowed energy levels of the harmonic oscillator from a knowledge of just ONE level

* $a_+$ is known as the CREATION or RAISING operator since it allows us to determine HIGHER levels from an initial energy solution

* $a_-$ is known as the ANNIHILATION or LOWERING operator since it allows us to determine LOWER levels from an initial energy solution




- *THE BASIC CONCEPT OF THE LADDER OPERATORS IS ILLUSTRATED SCHEMATICALLY IN THIS FIGURE*

- *THE IDEA IS THAT WE ASSUME THAT WE CAN START FROM SOME INITIAL KNOWN SOLUTION WITH ENERGY E*

- *SUCCESSIVE APPLICATION OF THE CREATION OPERATOR $a_+$ THEN ALLOWS US TO DETERMINE HIGHER ENERGY LEVELS WHILE APPLICATION OF THE ANNIHILATION OPERATOR ALLOWS US TO DETERMINE LOWER ENERGY LEVELS*

# The Operator Approach

Now the preceding discussion suggests that if we repeatedly apply the annihilation operator we should eventually reach a set of states with NEGATIVE energy!

* Since such states are physically MEANINGLESS there must instead be a MINIMUM state with positive energy that represents the lowest "rung" of our energy ladder
* That is we assume the existence of a wavefunction solution $y_0(x)$ such that

$$a_{-}\psi_0(x) = 0 \quad or \quad \frac{1}{\sqrt{2m}}\left[\frac{\hbar}{i}\frac{\partial}{\partial x} - im\omega x\right]\psi_0(x) = 0 \qquad (14.11)$$

* Expanding Equation 14.11 we arrive at the following condition on $y_0(x)$

$$\frac{\partial \psi_0(x)}{\partial x} = -\frac{m\omega}{\hbar} x \psi_0(x) \qquad (14.12)$$

# The Operator Approach

Equation 14.12 can be easily solved to yield the GROUND-STATE wavefunction

$$\int \frac{d\psi_0(x)}{\psi_0(x)} = -\frac{m\omega}{\hbar}\int x\,dx \quad \Rightarrow \quad \ln\psi_0(x) = -\frac{m\omega}{2\hbar}x^2$$

$$\therefore \quad \psi_0(x) = A_0 \exp\left[-\frac{m\omega}{2\hbar}x^2\right] \qquad (14.13)$$

* We now need to determine the ENERGY associated with this wavefunction which we do by substitution into the Schrödinger equation

$$\left[a_+a_- + \frac{1}{2}\hbar\omega\right]\psi_0(x) = E_0\psi_0(x) \qquad (14.14)$$

⇒ Since by definition $a_-\psi_0 = 0$ Equation 14.14 reduces to the SIMPLE result

$$\frac{1}{2}\hbar\omega\psi_0(x) = E_0\psi_0(x) \quad \therefore \quad E_0 = \frac{1}{2}\hbar\omega \qquad (14.14)$$

# The Operator Approach

Having determined the ground-state energy of the harmonic oscillator the creation operator can be used to generate the quantized SET of energy levels

$$E_n = \left[ n + \frac{1}{2} \right] \hbar\omega \,, \quad n = 0, 1, 2, \ldots \qquad (14.15)$$

* This result follows by remembering that each time we apply the creation operator we INCREASE the energy by h*w*

* The wavefunctions of the excited energy levels can also be reconstructed in the same way

$$\psi_n(x) = (a_+)^n \psi_0(x) = A_n (a_+)^n \exp\left[ -\frac{m\omega}{2\hbar} x^2 \right] \qquad (14.16)$$

⇒ These solutions are in fact IDENTICAL to those obtained previously (Equations 13.27 & 13.32) by analytic solution of the Schrödinger equation

# Examples

Determine the normalization constant $\psi_0(0)$ in the ground-state wavefunction

$$\psi_0(x) = A_0 \exp\left[-\frac{m\omega}{2\hbar}x^2\right] \qquad (13.13)$$

Solution: For normalization we require an expression for $\psi_0(0)$ such that

$$A_0^* A_0 \int_{-\infty}^{\infty} \exp\left[-\frac{m\omega}{\hbar}x^2\right]dx = 1$$

$\Rightarrow$ By exploiting the fact that

$$\int_{-\infty}^{\infty} \exp\left[-ax^2\right]dx = \sqrt{\pi / a}$$

$\Rightarrow$ We arrive at the following condition for the normalization constant

$$A_0^* A_0 \left[\frac{\pi\hbar}{m\omega}\right]^{1/2} = 1 \quad \therefore \quad A_0 = \left[\frac{m\omega}{\pi\hbar}\right]^{1/4}$$

# Examples

Use the following operator relation to determine the GENERAL normalization coefficient $A_n$ in Equation 14.16

$$a_+\psi_n(x) = i\sqrt{(n+1)\hbar\omega}\,\psi_{n+1}(x) \qquad (14.17)$$

Solution: We begin by making the change of variables $n \to n - 1$

$$a_+\psi_{n-1}(x) = i\sqrt{n\hbar\omega}\,\psi_n(x) \quad \therefore \quad \psi_n(x) = -\frac{i}{\sqrt{n\hbar\omega}}a_+\psi_{n-1}(x)$$

$\Rightarrow$ We can use this last relation to generate FURTHER wavefunction solutions

$$\psi_{n-1}(x) = -\frac{i}{\sqrt{(n-1)\hbar\omega}}a_+\psi_{n-2}(x) \qquad \psi_2(x) = -\frac{i}{\sqrt{2\hbar\omega}}a_+\psi_1(x)$$

$$\psi_{n-2}(x) = -\frac{i}{\sqrt{(n-2)\hbar\omega}}a_+\psi_{n-3}(x) \qquad \psi_1(x) = -\frac{i}{\sqrt{1\hbar\omega}}a_+\psi_0(x)$$

# Examples (cont'd)

By COMBINING the series of wavefunctions generated above we obtain an expression for $\psi_n(x)$ in terms of $\psi_0(x)$

$$\psi_n(x) = \frac{(-i)^n}{\sqrt{n!(\hbar\omega)^n}}(a_+)^n\psi_0(x) = \frac{(-i)^n}{\sqrt{n!(\hbar\omega)^n}}\left[\frac{m\omega}{\pi\hbar}\right]^{1/4}(a_+)^n \exp\left[-\frac{m\omega}{2\hbar}x^2\right] \qquad (14.18)$$

We have seen already however that the $n^{\text{th}}$ wavefunction can be written as

$$\psi_n(x) = A_n(a_+)^n \exp\left[-\frac{m\omega}{2\hbar}x^2\right] \qquad (14.16)$$

$\Rightarrow$ From a comparison of Equations 14.16 & 14.18 we see that the normalization constant

$$A_n = \frac{(-i)^n}{\sqrt{n!(\hbar\omega)^n}}\left[\frac{m\omega}{\pi\hbar}\right]^{1/4} \qquad (14.19)$$

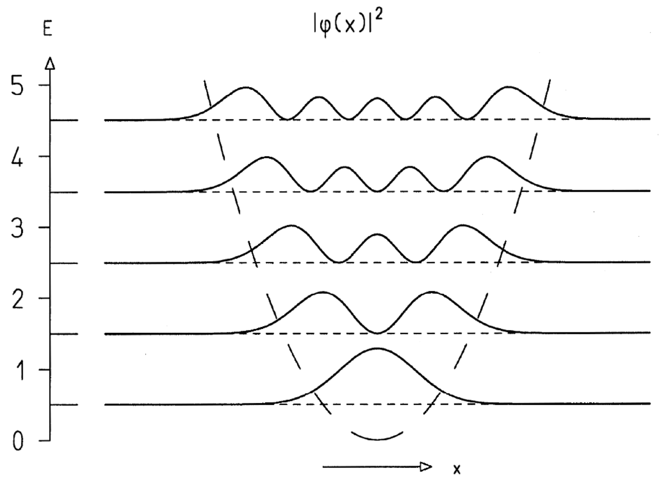
# Superposition States

Thus far we have focused on the solutions of the TIME-INDEPENDENT Schrödinger equation for a particle that moves in a HARMONIC potential

* The wavefunctions and probability density obtained for this problem take the form

$$\psi_n(x) = \left[\frac{m\omega}{\pi\hbar}\right]^{1/4} \frac{1}{\sqrt{2^n n!}} H_n\left[\sqrt{\frac{m\omega}{\hbar}}x\right] e^{-m\omega x^2/2\hbar} \qquad (13.32)$$

$$\psi_n^*(x)\psi_n(x) = \left[\frac{m\omega}{\pi\hbar}\right]^{1/2} \frac{1}{2^n n!} H_n^2\left[\sqrt{\frac{m\omega}{\hbar}}x\right] e^{-m\omega x^2/\hbar} \qquad (15.1)$$




- *THE PROBABILITY DENSITY ASSOCIATED WITH THE FIRST FIVE ENERGY LEVELS OF A HARMONIC OSCILLATOR*

- *THESE PROBABILITY DISTRIBUTIONS ARE INDEPENDENT OF TIME SINCE THEY DERIVE FROM WAVEFUNCTIONS THAT ARE STATIONARY STATES*

- *THE PICTURE BOOK OF QUANTUM MECHANICS S. BRANDT and H-D. DAHMEN, SPRINGER-VERLAG, NEW YORK (1995)*

# Superposition States

We now wish to discuss the DYNAMICS of a quantum-mechanical particle that moves in the presence of a harmonic potential

* We immediately run into a PROBLEM however since the wavefunctions of Equation 13.32 are STATIONARY STATES whose probability density is time INDEPENDENT

⇒ Recall that the stationary states have time-dependent wavefunctions of the form

$$\Psi_n(x,t) = \psi_n(x,0)\exp\left[-i\frac{E_n}{\hbar}t\right] \qquad (15.2)$$

⇒ The probability density associated with this wavefunction is easily calculated

$$\Psi_n^*(x,t)\Psi_n(x,t) = \psi_n^*(x,0)\exp\left[+i\frac{E_n}{\hbar}t\right]\psi_n(x,0)\exp\left[-i\frac{E_n}{\hbar}t\right]$$
$$= \psi_n^*(x,0)\psi(x,0) \qquad (15.3)$$

***THE PROBABILITY DENSITY IS INDEPENDENT OF TIME!***

# Superposition States

To overcome this problem we note that a more GENERAL solution to the Schrödinger equation is obtained by taking a LINEAR SUPERPOSITION of stationary-state wavefunctions

$$\psi(x) = \sum_n c_n \psi_n(x) \qquad (15.3)$$

* The TIME EVOLUTION of this wavefunction is now given by

$$\Psi(x,t) = \sum_n c_n \psi_n(x) \exp\left[-i\frac{E_n}{\hbar}t\right] \qquad (15.4)$$

* As we will discuss in further detail an important property of the stationary state wavefunctions is that they define a so-called ORTHONORMAL SET

⇒ The condition of orthonormality can be expressed as

$$\int_{-\infty}^{\infty} \psi_n^*(x)\psi_k(x)dx = \delta_{nk}, \quad \delta_{nk} = \begin{cases} 1, & n = k \\ 0, & n \neq k \end{cases} \qquad (15.5)$$

***$\delta_{nk}$ IS THE KRONECKER DELTA FUNCTION***

# Superposition States

By exploiting the orthonormality of the stationary-state solutions the expansion coefficients in Equation 15.3 can be determined

$$\int_{-\infty}^{\infty} \psi_n^*(x)\psi(x)dx = \int_{-\infty}^{\infty} \psi_n^*(x)\sum_n c_n\psi_n(x)dx$$

$$= c_n \int_{-\infty}^{\infty} \psi_n^*(x)\psi_n(x)dx = c_n \qquad (15.6)$$

* Another important result follows from the NORMALIZATION condition for the wavefunction

$$\int_{-\infty}^{\infty} \psi^*(x)\psi(x)dx = 1 \quad \Rightarrow \quad \int_{-\infty}^{\infty} \sum_n c_n^*\psi_n^*(x)\sum_n c_n\psi_n(x)dx = 1$$

$$\therefore \quad \sum_n c_n^* c_n \int_{-\infty}^{\infty} \psi_n^*(x)\psi_n(x)dx = 1 \quad \Rightarrow \quad \sum_n c_n^* c_n = 1 \qquad (15.7)$$

***NORMALIZATION RELATION FOR THE EXPANSION COEFFICIENTS***

# Harmonic Particle Motion

An important feature of the linear superposition of stationary states is that it results in a WAVE PACKET whose properties now EVOLVE with time

* Consider for example the expectation value of the POSITION of the wave packet

$$< x(t) > = (\Psi(x,t), x\Psi(x,t)) \equiv \int_{-\infty}^{+\infty} \Psi^*(x,t) x \Psi(x,t)\, dx$$

$\Rightarrow$ By introducing Equation 15.3 for the wavefunction this expectation may be written as

$$< x(t) > = \int_{-\infty}^{\infty} \sum_n c_n^* \psi_n^*(x) e^{+iE_n t/\hbar} \sum_k x c_k \psi_k(x) e^{-iE_k t/\hbar} dx$$

$$= \sum_n \sum_k c_n^* c_k e^{+i(E_n - E_k)t/\hbar} \int_{-\infty}^{\infty} \psi_n^*(x) x \psi_k(x) dx$$

$$\equiv \sum_n \sum_k c_n^* c_k e^{+i(E_n - E_k)t/\hbar} x_{nk} \qquad (15.8)$$

***NOTE HOW WE DEFINE THE MATRIX ELEMENT $x_{nk}$***

# Harmonic Particle Motion

• To compute the expectation value of the position we now have to evaluate the MATRIX ELEMENT $x_{nk}$ that appears in Equation 15.8

* By exploiting the properties of Hermite polynomials (see Appendix) it can be shown that the matrix element reduces to

$$x_{nk} = \sqrt{\frac{\hbar}{m\omega}}\left[\sqrt{\frac{n}{2}}\delta_{k,n-1} + \sqrt{\frac{n+1}{2}}\delta_{k,n+1}\right] \qquad (15.9)$$

* Substituting this expression into Equation 15.8 we arrive at the following result for the expectation value of the position

$$< x(t) > = \sqrt{\frac{\hbar}{2m\omega}}\sum_n n^{1/2}\,(c_n^* c_{n-1} e^{i\omega t} + c_{n-1}^* c_n e^{-i\omega t}) \qquad (15.10)$$

$\Rightarrow$ We have exploited here the fact that $E_n - E_k = (n - k)\hbar\omega$ so that $E_n - E_{n-1} = \hbar\omega$

# Harmonic Particle Motion

Since the expansion coefficients $c_n$ may be COMPLEX quantities we can write them in POLAR form

$$c_n = |c_n| e^{i\phi_n} \qquad (15.11)$$

* With this definition we can then rewrite Equation 15.10 as

$$< x(t) > = \sqrt{\frac{2}{m\omega^2}} \sum_n \sqrt{n\hbar\omega}\, c^*_{n-1} c_{n-1}\, c^*_n c_n \cos(\omega t + \phi_{n-1} - \phi_n) \qquad (15.12)$$

* If the PHASE ANGLE $\phi_n - \phi_{n-1} = \alpha$ where $\alpha$ is a CONSTANT independent of $n$ and if the expansion coefficients $c_n$ are of EQUAL magnitude then Equation 15.12 becomes

$$< x(t) > \approx \sqrt{\frac{2}{m\omega^2}} \left[ \sum_n \sqrt{E_n}\, c^*_n c_n \right] \cos(\omega t + \alpha) = x_o \cos(\omega t + \alpha) \qquad (15.13)$$

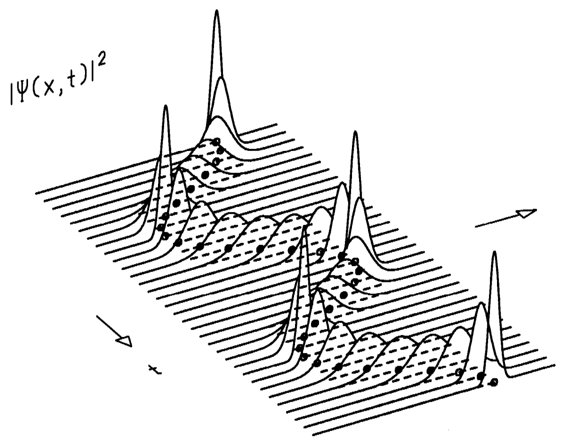
***THIS TERM IS THE EXPECTATION VALUE OF $\sqrt{E}$***

# Harmonic Particle Motion

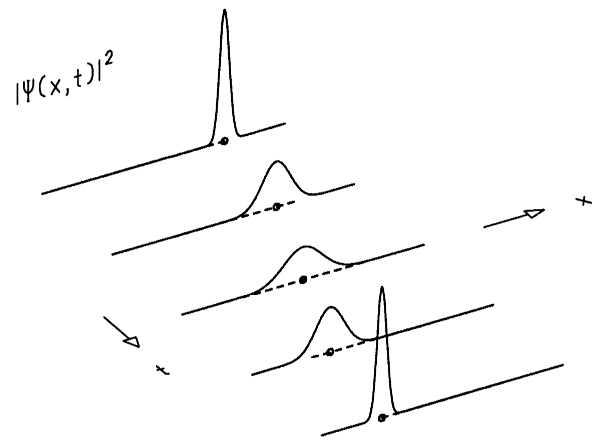
Equation 15.13 shows that the expectation value of the wavepacket in the parabolic potential OSCILLATES as a function of time

* The oscillations occur at the SAME frequency ($\omega$) of the classical oscillator
* From the figures below we see that the WIDTH of the wavepacket also oscillates but at TWICE the frequency of the classical oscillator

$\Rightarrow$ The width of the packet is narrowest at the classical TURNING POINTS where the velocity of the classical particle is zero




***MOTION OF A WAVE PACKET IN A HARMONIC POTENTIAL CIRCLES SHOW THE MOTION OF A CLASSICAL PARTICLE***




***THE SAME MOTION IS SHOWN IN MORE DETAIL OVER HALF AN OSCILLATION CYCLE***

***THE PICTURE BOOK OF QUANTUM MECHANICS S. BRANDT and H-D. DAHMEN, SPRINGER-VERLAG, NEW YORK (1995)***

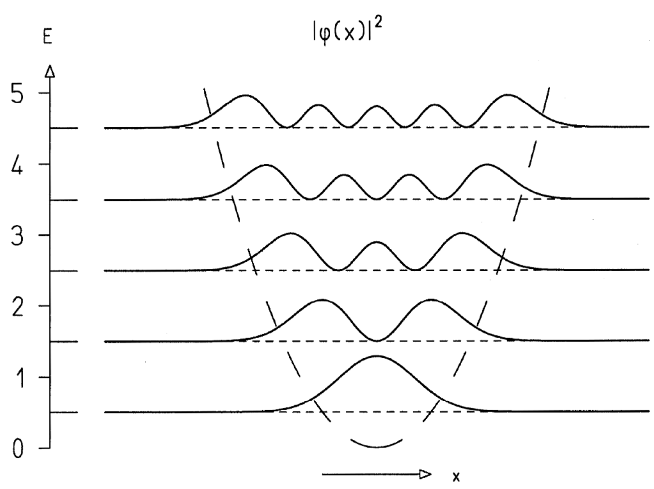
# Harmonic Particle Motion

Instead of a wavepacket what is the corresponding variation of the position expectation in one of the STATIONARY states?

* For such a state ALL coefficients but one are equal to zero so that Equation 15.10 reduces to

$$< x(t) > = \sqrt{\frac{\hbar}{2m\omega}} \sum_n n^{1/2} (c_n^* c_{n-1} e^{i\omega t} + c_{n-1}^* c_n e^{-i\omega t}) = 0 \qquad (15.14)$$

$\Rightarrow$ The expectation value is ZERO for ALL times as we expect for the stationary states which have SYMMETRIC probability densities




- *THE PROBABILITY DENSITY ASSOCIATED WITH THE FIRST FIVE ENERGY LEVELS OF A HARMONIC OSCILLATOR*
- *NOTE THAT THESE PROBABILITY DENSITIES ARE ALL SYMMETRIC ABOUT THE ORIGIN OF MOTION AND SO GIVE RISE TO AN EXPECTATION VALUE FOR THE POSITION THAT IS EQUAL TO ZERO AT ALL TIMES*
- *THE PICTURE BOOK OF QUANTUM MECHANICS S. BRANDT and H-D. DAHMEN, SPRINGER-VERLAG, NEW YORK (1995)*

# Harmonic Particle Motion

The other quantity in which we are interested is the expectation value of the MOMENTUM

* The SIMPLEST way to determine this quantity is to note that we expect the following relation to hold

$$< p(t) > = m < v(t) > = m\frac{d}{dt} < x(t) > \qquad (15.15)$$

* By differentiating Equation 15.10 with respect with time we obtain the following result

$$< p(t) > = i\sqrt{\frac{\hbar m\omega}{2}}\sum_n n^{1/2}\,(c_n^* c_{n-1} e^{i\omega t} - c_{n-1}^* c_n e^{-i\omega t}) \qquad (15.10)$$

⇒ The time dependence of this equation is IDENTICAL to that of Equation 15.10

⇒ We therefore expect that the expectation value of the momentum should also OSCILLATE as a function of time

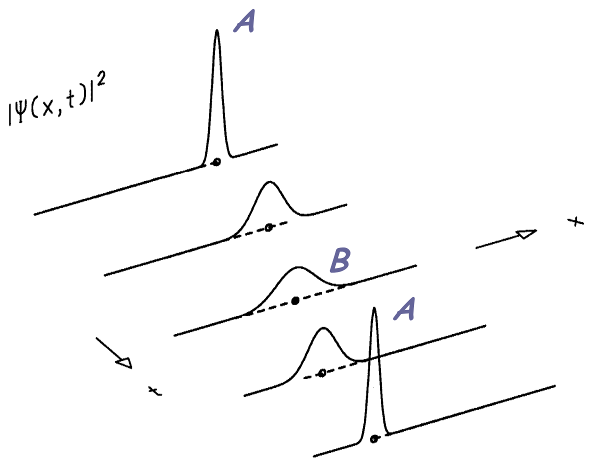
# Harmonic Particle Motion

Under conditions where the assumptions that lead to Equation 15.13 hold the expectation value of the momentum can be APPROXIMATED as

$$< p(t) > \approx \sqrt{2m}\left[\sum_n \sqrt{E_n} c_n^* c_n\right] \sin(\omega t + \alpha) = p_o \sin(\omega t + \alpha) \qquad (15.13)$$

* Note that the oscillations of the position and momentum are OUT of phase

⇒ This is just what we EXPECT for the harmonic oscillator which is at REST when its displacement is MAXIMAL and moves FASTEST when its displacement is ZERO




- *THE TWO POINTS LABELED A CORRESPOND TO THE MAXIMUM DISPLACEMENT OF THE HARMONIC OSCILLATOR AND AT THESE POINTS THE PARTICLE IS INSTANTANEOUSLY STATIONARY*
- *THE POINT LABELED B CORRESPONDS TO THE CENTER OF THE HARMONIC MOTION WHERE THE DISPLACEMENT OF THE PARTICLE IS EQUAL TO ZERO*
- *THE PARTICLE IS MOVING WITH ITS HIGHEST VELOCITY HERE AND SO THE EXPECTATION VALUE OF THE MOMENTUM IS CONSEQUENTLY MAXIMAL*

# Appendix

In this section we consider how to evaluate the MATRIX ELEMENTS

$$x_{nk} = \int_{-\infty}^{\infty} \psi_n^*(x)\, x\, \psi_k(x)\, dx \qquad (A15.1)$$

* As we have seen the wavefunctions $y_n(x)$ correspond to HERMITE POLYNOMIALS

$$\psi_n(x) = \left[\frac{m\omega}{\pi\hbar}\right]^{1/4} \frac{1}{\sqrt{2^n n!}} H_n(\zeta) e^{-\zeta^2/2}, \quad \zeta \equiv \sqrt{\frac{m\omega}{\hbar}}\, x \qquad (13.32)$$

* So that Equation A15.1 can be rewritten as

$$x_{nk} = \sqrt{\frac{1}{2^{n+k}\pi n! k!}}\, \frac{\hbar}{m\omega} \int_{-\infty}^{\infty} H_n(\zeta) H_k(\zeta) \zeta e^{-\zeta^2}\, d\zeta \qquad (A15.2)$$

# Appendix

To evaluate the integral in Equation A15.2 shall need our DEFINITIONS of the generating function

$$F(s,\zeta) = \sum_{n=0}^{\infty} H_n(\zeta)\frac{s^n}{n!} \qquad (A13.2)$$

$$F(s,\zeta) = e^{-s^2} e^{-2s\zeta} = e^{\zeta^2-(s-\zeta)^2} \qquad (A13.9)$$

* Rather than compute Equation A15.2 directly we first construct the more GENERAL integral

$$I = \int_{-\infty}^{\infty} F(s,\zeta)F(t,\zeta)e^{2\lambda\zeta-\zeta^2}d\zeta \qquad (A15.3)$$

* Equation A15.3 may be rewritten using Equations A13.2 & A13.9 as

$$\int_{-\infty}^{\infty} e^{\zeta^2-(s-\zeta)^2} e^{\zeta^2-(t-\zeta)^2}\, e^{2\lambda\zeta-\zeta^2} d\zeta = \sum_n \sum_k \frac{s^n t^k}{n!k!}\int_{-\infty}^{\infty} H_n(\zeta)H_k(\zeta)e^{2\lambda\zeta-\zeta^2}d\zeta \qquad (A15.4)$$

# Appendix

The LHS of Equation A15.4 can be evaluated explicitly to yield

$$e^{2st+\lambda^2+2\lambda(s+t)}\int_{-\infty}^{\infty} e^{-(s+t+\lambda-\zeta)^2}d\zeta = \sqrt{\pi}\, e^{\lambda^2+2(st+\lambda s+\lambda t)} \qquad (A15.5)$$

* Comparing the coefficients of equal powers of $s^n t^k l^p$ we obtain the value of a useful integral

$$\int_{-\infty}^{\infty} H_n(\zeta)H_k(\zeta)e^{-\zeta^2}\zeta^p d\zeta \qquad (A15.6)$$

* For example the integral in Equation A15.2 is obtained by setting $p = 1$ which yields

$$\int_{-\infty}^{\infty} H_n(\zeta)H_k(\zeta)e^{-\zeta^2}\zeta d\zeta = \sqrt{\pi}\, 2^{n-1} n!(\delta_{k,n-1} + 2(n+1)\delta_{k,n-1}) \qquad (A15.7)$$

$\Rightarrow$ Using Equation A15.7 the matrix element of Equation A15.2 can be computed and hence the expectation value of the position can be determined (Equation 15.10)