Objectives

- Develop a set of tools useful throughout the course
2.1 Linear Systems

- Consider a simple system:
- Equation of motion:

\[ m \frac{d^2 x}{dt^2} + \gamma m \frac{dx}{dt} + m \omega_0^2 x = f(t) \]  \hspace{1cm} (2.1)

- Define Operator: (linear differential eqs)

\[ L = m \frac{d}{dt} \left( \frac{d}{dt} \right) + \gamma m \frac{d}{dt} + m \omega_0^2 \]  \hspace{1cm} (2.2)

\[ \Rightarrow L(x) = F(t) \]  \hspace{1cm} (2.3)
2.1 Linear Systems

- Operator L has important properties:
  
  a) \( L(ax) = m \frac{d(ax)}{dt^2} + \gamma m \frac{d(ax)}{dt} + m\omega_o^2(ax) = \)
  
  \[
  = a \left[ m \frac{d(x)}{dt^2} + \gamma m \frac{d(x)}{dt} + m\omega_o^2(x) \right] = 
  \]
  
  \[
  = aL(x) \tag{2.4}
  \]

  b) \( L(x + y) = m \frac{d(x + y)}{dt^2} + \gamma m \frac{d(x + y)}{dt} + m\omega_o^2(x + y) = \)
  
  \[
  = L(x) + L(y) \tag{2.5}
  \]
2.1 Linear Systems

- **Definition**: An operator obeying properties $L(ax) = aL(x)$ and $L(x+y) = L(x)+L(y)$ is called **linear**
- Most of the system in nature are linear; well, at least to the first approximation
- They are mathematically tractable $\rightarrow$ **analytic solutions**
- Consider equations:

$$
\begin{align*}
L(x_1) &= 0 \\
L(x_2) &= 0
\end{align*}
$$

$(2.6)$

$\rightarrow x_1, x_2$ are solutions
Continuing:

\[ L(ax_1 + bx_2) = L(ax_1) + L(bx_2) = aL(x_1) + bL(x_2) = 0 + 0 \]  \hspace{1cm} (2.7)

Any linear combination of solutions: \( x_1, x_2 \) is also a solution

The number of independent solutions = degrees of freedom

\[ X_1, X_2, \ldots, X_N = \text{independent solutions if} \]

\[ X_i \neq \sum_{j \neq i} \alpha_j x_j, \text{ for any } \alpha_j \] \hspace{1cm} (2.8)

Linear Differential eqs of order N allow for N independent solutions
2.2 Light-matter interaction

- Classic model of atom: $e^-$ rotating around $N \approx$ planets

$\rightarrow$ Lorentz Model

- Analogy:

$$x = x_0 \cos(\omega_0 t)$$

$$\omega = \sqrt{\frac{k}{m}}$$
2.2 Light-matter interaction

- So, notion of charge follows the same eq (2.1)

\[ m \frac{d^2x}{dt^2} + \gamma m \frac{dx}{dt} + m\omega_o^2 x = F(t) \]

- Incident field drives the charge: \( \bar{F}(t) = q\bar{E}(t) \) (2.9)

- For e\(^-\), q = -e

- Monochromatic field: \( E(t) = E_o e^{-i\omega t} \)

\[ m\ddot{x} + \gamma m\dot{x} + m\omega_o^2 = qE_o e^{-i\omega t} \] (2.10)

- This is the eq of motion for eletric charge under incident EM field. Can explain most of Optics!
2.3 Superposition principle

- Suppose we have 2 fields simultaneously interacting with the material (Eg. \( \omega_1, \omega_2 \)):

\[
E_1 = |E_1|e^{-i\omega_1 t}; qE_1 = F_1 \\
E_2 = |E_2|e^{-i\omega_2 t}; qE_2 = F_2
\]  \hspace{1cm} (2.11)

- Let \( x_1, x_2 \) be solutions of displacements for the two forces \( F_1 \) and \( F_2 \)

\[
\begin{cases}
L(x_1) = F_1(t) \\
L(x_2) = F_2(t)
\end{cases}
\]  \hspace{1cm} (2.12)
2.3 Superposition principle

- Consider the same solution:

\[
L(x_1 + x_2) = L(x_1) + L(x_2) = F_1(t) + F_2(t)
\]  

(2.13)

- So, final solution is just the sum of individual solutions. Nice!

- This is the superposition principle

- For the 2 frequency example:

- It’s as if one applies the fields one by one and sums their results
2.4 Green’s function/impulse response

- Let the incident field, i.e. driving field, have a complicated shape

\[ \delta(t) = \begin{cases} 
0, & t = 0 \\
1, & \text{otherwise} 
\end{cases} \] \hspace{1cm} (2.14)

\[ E(t) = \int_{\infty}^{\infty} E(t') \delta(t - t') dt' \] \hspace{1cm} (2.15)

- \( E(t) \) can be broken down into a succession of short pulses, i.e. Dirac delta functions:
2.4 Green’s function/impulse response

- If we know the response of the system to a short pulse, $\delta(t)$, the problem is solved.
- Let $h(t)$ be the solution to $\delta(t)$.
- The final solution for an arbitrary force $\vec{F}(t) = q\vec{E}(t)$ is:

$$x(t) = \int_{-\infty}^{\infty} E(t') h(t - t') dt'$$  \hspace{1cm} (2.16)

- This is the Green’s method of solving linear problems.
- $h(t) =$ Green’s function or impulse response of the system.
- Complicated problems become easily tractable!
2.5 Fourier Transforms

- Very efficient tool for analyzing linear (and non-linear) processes

- **Definition:**
  \[
  \mathcal{F}[f(x)] = \int_{-\infty}^{\infty} f(x)e^{-i2\pi fx} \, dx
  \]
  
  \[
  = F(f_x) = \tilde{f}(\xi)
  \]

- F is the Fourier transform of f

- \( f : \Delta \to \Delta; \Delta \in \mathbb{C} \), f must satisfy:
  
  a) \( \int |f| < \infty \) - modulus integrable
  
  b) f has finite number of discontinuities in the finite domain \( \Delta \)
  
  c) f has no infinite discontinuities

- In practice, some of these conditions are sometimes relaxed
2.5 Fourier Transforms

- **Inverse Fourier Transforms:**

\[
\mathcal{F}^{-1}\left[\mathcal{F}(f(x))\right] = \int_{-\infty}^{\infty} \tilde{f}(\xi)e^{+i2\pi f_x} df_x
\]

\[
= f(x)
\]

\[
\Rightarrow \quad \mathcal{F}^{-1}[\mathcal{F}(f)] = f
\]

- **Meaning of F.T:** reconstruct a complicated signal by summing sinusoidals with proper weighting

\[\text{(2.18)}\]

\[\text{(2.19)}\]
2.5 Fourier Transforms

- Fourier transform is a **linear operator**:

\[
\mathcal{F}[af(x) + bg(x)] = \\
\quad = \int_{-\infty}^{\infty} [af(x) + bg(x)]e^{-i2\pi \xi x} dx = \\
\quad = a\int_{-\infty}^{\infty} f(x)e^{-i2\pi \xi x} dx + b\int_{-\infty}^{\infty} g(x)e^{-i2\pi \xi x} dx \\
\quad = a\mathcal{F}[f(x)] + b\mathcal{F}[g(x)]
\] (2.20)
2.6 Basic Theorems with Fourier Transforms

a) **Shift Theorem:** if $\tilde{f}(\xi) = \mathbb{F}[f(x)]$

\[
\mathbb{F}\{f(x - a)\} = \tilde{f}(\xi)e^{-i2\pi\xi a}
\]

(2.21)

- Easy to prove using definition
- Eq 2.21 suggest that a shift in one domain corresponds to a linear phase ramp in the other (Fourier) domain
2.6 Basic Theorems with Fourier Transforms

b) Parseval’s theorem: if $\mathcal{F}[f(x)] = \tilde{f}(\xi)$

$$\int_{-\infty}^{\infty} |f(x)|^2 \, dx = \int_{-\infty}^{\infty} |\tilde{f}(\xi)|^2 \, d\xi$$  \hspace{1cm} (2.22)

- Conservation of total energy
2.6 Basic Theorems with Fourier Transforms

c) **Similarity theorem:** if

\[ \mathcal{F}[f(x)] = \tilde{f}(f_x) \], i.e. \( \tilde{f} \) is the F.T of \( f \)

\[
\mathcal{F}[f(ax)] = \frac{1}{|a|} \tilde{f}\left(\frac{\xi}{a}\right)
\]  \hspace{1cm} (2.23)

- Theorem 2.23 provides intuitive feeling for F.T
- Let’s consider
2.6 Basic Theorems with Fourier Transforms

c) Similarity theorem:
- Let’s consider:

\[ f(x) \rightarrow \mathcal{F} \rightarrow \tilde{f}(f_x) \]

\[ f(x) \rightarrow \mathcal{F} \rightarrow \tilde{f}(f_x) \]

\[ f(x) \rightarrow \mathcal{F} \rightarrow \tilde{f}(f_x) \]

\[ f(x) \rightarrow \mathcal{F} \rightarrow \tilde{f}(f_x) \]
2.6 Basic Theorems with Fourier Transforms

c) **Similarity theorem:**
- Broader functions in one domain implies narrower functions in the other domain and vice-versa.
- Eg. To obtain short **temporal** pulses of light, one needs a broad spectrum (Ti: Saph laser).
- Only an infinite spectrum allows for $\delta$-functions pulses.

$\delta$ functions pulses

**Physically Impossible**
Before we present the last theorems, we introduce the definitions of convolution and correlation.

Let

\[ g(x) \xrightarrow{\mathcal{F}} G(\xi) \]
\[ h(x) \xrightarrow{\mathcal{F}} H(\xi) \]

**Convolution of \( g \) and \( h \):**

\[ g \odot h = \int_{-\infty}^{\infty} g(x')h(x-x')dx' \tag{2.24} \]

**Correlation of \( g \) and \( h \)**

\[ g \otimes h = \int_{-\infty}^{\infty} g(x')h(x'-x)dx' \tag{2.25} \]
2.6 Basic Theorems with Fourier Transforms

- Difference between $\otimes$ and $\lor$ is $h(x-x')$ vs $h(x'-x)$, i.e. flip vs non-flip of $h$

- Particular case:
  - **Autocorrelation**: $g=h$
    \[
    g \otimes g = \int_{-\infty}^{\infty} g(x')g(x'-x)dx'
    \]  
    (2.26)

- **Exercise**: Use PC to show:
  - $\cdots \otimes \cdots = \cdots$
  - $\cdots \otimes \cdots = \cdots$
  - Gauss $\otimes$ Gauss = Gauss
2.6 Basic Theorems with Fourier Transforms

d) Convolution theorem:
\[ \mathcal{F}[g \ast h] = GH \]  \hspace{1cm} (2.27)

\[ \text{i.e. } \mathcal{F} \left[ \int_{-\infty}^{\infty} g(x')h(x-x')dx' \right] = G(\xi)H(\xi) \]

- Convolution in one domain corresponds to a product in the other. Nice!
- Multiplication is always easy to do
- Recall Green’s function: \( h(t) = \) the response to a \( \delta \)-function light pulse
2.6 Basic Theorems with Fourier Transforms

- We found (Eq 2.16):

\[ x(t) = \int_{-\infty}^{\infty} E(t') h(t - t') dt' \]

i.e. the response to an arbitrary field \( E(t) \) is the convolution \( E \ast h \)!

- Let’s take the F.T:

\[ x(\omega) = E(\omega) h(\omega) \quad (2.28) \]

\( \rightarrow \) It doesn’t get any simpler than this

i.e. if we know the impulse response \( h(t) \), (or the Green’s function) take F.T \( \rightarrow h(\omega) \equiv \text{transfer function} \rightarrow \) response to any field \( E \) is:

\[ x(t) = \mathcal{F}[E(\omega) h(\omega)] \quad (2.29) \]
2.6 Basic Theorems with Fourier Transforms

e) Correlation theorem:

- $\otimes$ differs from $\odot$ only by minus sign $\implies$ similar theorem:

$$\mathcal{F}[g \otimes h] = GH^*$$

(2.30)

i.e

$$\mathcal{F} \left[ \int_{-\infty}^{\infty} g(x')h(x' - x)dx' \right] = G(\xi)H(\xi)^*$$

$\implies$ Particular case: $g = h$ (auto correlation):

$$\mathcal{F}[g \otimes g] = GG^* = |G|^2$$

(2.31)
2.6 Basic Theorems with Fourier Transforms

e) Correlation theorem:

- **Eg**: F.T of an auto correlation is the power spectrum
- Very important for both time and space fluctuating fields:

\[
\Gamma(t) = \int_{-\infty}^{\infty} E(t')E(t'-t)dt = \text{auto correlation}
\]

\[
\mathcal{F}\{\Gamma(t)\} = E(\omega)E^*(\omega) = S(\omega) = \text{power spectrum}
\]

(Wiener–Khinchin theorem)

- We’ll meet them again later!
2.7 Differential equations and Fourier Transforms

- Let \( f \) be a function of time:

\[
f(t) = \int_{-\infty}^{\infty} F(\omega) e^{i\omega t} d\omega = \mathcal{F}^{-1}(F)
\]

(2.33)

- What is \( \frac{\partial f}{\partial t} \)?

\[
\frac{\partial f}{\partial t} = \frac{\partial}{\partial t} \left[ \int_{-\infty}^{\infty} F(\omega) e^{i\omega t} d\omega \right] = \\
= \int_{-\infty}^{\infty} F(\omega) \frac{\partial}{\partial t} [e^{i\omega t}] d\omega = \\
= \int_{-\infty}^{\infty} [i\omega F(\omega)] e^{i\omega t} d\omega
\]

(2.34)

- So, \( f \rightarrow F \) & \( \frac{\partial f}{\partial t} \rightarrow i\omega F \)
2.7 Differential equations and Fourier Transforms

- **Great:**
  \[
  \mathcal{F}[f(t)] = F(\omega). \text{ Then:} \]
  \[
  \mathcal{F}\left[\frac{\partial f(t)}{\partial t}\right] = i\omega F(\omega) \rightarrow \text{useful} \tag{2.34}
  \]

- **Now**
  \[
  \mathcal{F}\left[\frac{\partial^2 f}{\partial t^2}\right] = \mathcal{F}\left[\frac{\partial}{\partial t}\left(\frac{\partial f}{\partial t}\right)\right] = i\omega i\omega F(\omega)
  \]
  \[
  = -\omega^2 F(\omega)
  \]
  In others words:
  \[
  \mathcal{F}\left[\frac{\partial^n f}{\partial t^n}\right] = i^n \omega^n F(\omega) \tag{2.35}
  \]

- **Differentiation theorem**
Why 2.35 result is important? Because linear differential equations are resolved in the frequency domain more easily.

**Eg**: Recall our e\(^{-}\) revolving around nucleus under field illumination \(E(t)\)

\[
m \frac{d^2 x(t)}{dt^2} + \gamma m \frac{dx(t)}{dt} + m\omega_o^2 x(t) = qE(t)
\]  

(2.36)

The solution is \(x(t)\). But we can solve for \(x(\omega) = \mathcal{F}[x(t)]\) and take \(\mathcal{F}^{-1}\) in the end.
2.7 Differential equations and Fourier Transforms

- So, let’s take F.T of 2.36, using the differentiation theorem:

\[ m[-\omega^2 x(\omega)] + i\omega\gamma m x(\omega) + m\omega_o^2 x(\omega) = qE(\omega) \]

\[ x(\omega)[-m\omega^2 + i\omega\gamma m + m\omega_o^2] = qE(\omega) \]

Since \( q = -e \):

\[
x(\omega) = \frac{e}{m} \frac{E(\omega)}{-\omega^2 - i\gamma\omega - \omega_o^2}
\]

(2.37)
2.7 Differential equations and Fourier Transforms

- **Exercise**: use PC to take $\mathcal{F}^{-1}$ of 2.37

- “damped” oscillation, $\gamma = $ damping factor
- !Problem solved
Given the electron displacement as a function of frequency, $X(\omega)$, we can define the dipole moment:

$$\overline{p} = q\bar{x}$$

$$p = -ex$$  \hspace{1cm} (2.38)

The dipole moment is a **microscopic** quantity; we need a **macroscopy** counterpart:

$$\overline{P} = N \langle \overline{p} \rangle = -\frac{Ne^2}{m} \cdot E$$

$$\frac{1}{\omega^2 - \omega_o^2 - i\gamma\omega}$$  \hspace{1cm} (2.39)
2.8 Refraction and absorption. Dispersion

- \( \vec{P} \equiv \text{induced polarization} \)
- \( \vec{N} \equiv \text{concentration} \) [m\(^{-3}\)]
- But \( \vec{P} \) relates to the macroscopic response of the material \( X \), i.e. electric susceptibility:

\[
\vec{P} = \varepsilon_0 \chi \vec{E}
\]  \hspace{1cm} (2.40)

- \( \varepsilon_0 \) = permeability of vacuum
- Finally, \( \chi = \varepsilon_r - 1 = n^2 - 1 \) \hspace{1cm} (2.41)
- \( \varepsilon_r \) = relative permeability
- \( n = \text{refractive index} \)
- If \( \chi \in \mathbb{R} \), as opposed to \( \chi = \begin{pmatrix} \chi_{11} & \chi_{12} & \chi_{13} \\ \chi_{21} & \chi_{22} & \chi_{23} \\ \chi_{31} & \chi_{32} & \chi_{33} \end{pmatrix} \), material is isotropic

Chapter 2: Math Toolbox
2.8 Refraction and absorption. Dispersion

- So, combining 2.39 and 2.40:

\[ \chi = \frac{Ne^2 / m\varepsilon_o}{(\omega^2 - \omega_o^2) - i\gamma\omega} = n^2 - 1 \in \mathbb{C} \quad (2.42) \]

- For low-n materials, such as rarefied gases,

\[ n^2 - 1 = (n - 1)(n + 1) \approx 2(n - 1) \]

\[ n = 1 + \frac{Ne^2}{2m\varepsilon_o} \frac{1}{(\omega^2 - \omega_o^2) - i\gamma\omega} \]

\[ = n' + in'' \quad (2.43) \]
2.8 Refraction and absorption. Dispersion

\[
\begin{align*}
n' &= 1 + \frac{N e^2}{2 m \varepsilon_0} \frac{\omega^2 - \omega_o^2}{(\omega^2 - \omega_o^2) - \gamma^2 \omega^2} \\
n'' &= \frac{N e^2}{2 m \varepsilon_0} \frac{\gamma \omega}{(\omega^2 - \omega_o^2) - \gamma^2 \omega^2}
\end{align*}
\]  

(2.44a) (2.44b)

- \( n' = \text{Re}(n) = \text{refractive index} \)
- \( n'' = \text{Im}(n) = \text{absorption index} \)
2.8 Refraction and absorption. Dispersion

- **Eg:** Plane wave:

\[
E = E_o e^{ik \cdot r} ; k = nk_o
\]

\[
E = E_o e^{ink_o r} = E_o e^{ik_o r (n' + in'')}
\]

\[
E = E_o e^{-n'^1k_o r} e^{in'k_o r}
\]

- absorption
- refraction

\[
\alpha = n'^1k_o = \text{absorption coefficient}
\]
2.8 Refraction and absorption. Dispersion

- **Definition:**
  
  \[ n'(\omega) = \text{variation of refractive index with frequency} \]
  
  \[ \text{= dispersion} \]
  
  \[ n''(\omega) = \text{absorption line shape} \]

![Diagram showing the variation of refractive index with frequency and absorption line shape]
2.8 Refraction and absorption. Dispersion

- Note the line shape:

\[
\frac{\gamma \omega}{(\omega^2 - \omega_o^2) + \gamma^2 \omega^2} \approx \frac{1}{\gamma \omega}\left[1 + \left(\frac{(\omega - \omega_o)2\omega_o}{\gamma \omega}\right)^2\right] = \frac{1}{\gamma \omega}\left[1 + \left(\frac{\omega - \omega_o}{\gamma \omega / 2\omega_o}\right)^2\right]
\]

- Lorentz function:

\[
\tilde{\mathcal{S}}(\omega) = \frac{1}{a}\left(\frac{1}{1 + (\omega / a)^2}\right); \quad a = \text{width}
\]
2.8 Refraction and absorption. Dispersion

- Thus the absorption line is a Lorentzian:

\[ \alpha(\omega; \omega_o) = \frac{1}{\Delta \omega} \frac{1}{1 + \left( \frac{\omega - \omega_o}{\Delta \omega} \right)^2}; \Delta \omega \sim \gamma \]

\[ \Im[\alpha(\omega)] = e^{-\Delta \omega |t|} \]

- The Fourier transform of a Lorentzian is an exponential!
Connect to quantum mechanics:

- 2 level system:
  \[ \begin{align*}
  1 & \rightarrow E_1 \\
  0 & \rightarrow E_0 \\
  \end{align*} \]
  \[ \Delta E = E_1 - E_o = \hbar(\omega_1 - \omega_o) \]

- Probability of spontaneous emission/absorption:
  \[ p(t) \sim e^{-t/t\text{lifetime}} \rightarrow \text{exponential decay} \]

- Linewidth is Lorentz = natural linewidth

- The model of $e^-$ on springs was introduced by Lorentz
2.9 Maxwell’s Equations

- Fully describe the propagation of EM fields
- Quantify how $\hat{E}$ and $\hat{H}$ generate each other

\[
\begin{align*}
\nabla \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t} \quad &\text{I} \\
\nabla \times \vec{H} &= \frac{\partial \vec{D}}{\partial t} + \vec{j} \quad &\text{II} \tag{2.45a} \\
\n\nabla \vec{D} &= \rho \quad &\text{III} \\
\n\nabla \vec{B} &= 0 \quad &\text{IV}
\end{align*}
\]
2.9 Maxwell’s Equations

- Plus material equations

\[
\begin{align*}
\vec{D} &= \varepsilon_0 \vec{E} + \vec{P} \\
\vec{B} &= \mu_0 \vec{H} + \vec{M}
\end{align*}
\]

- Definitions:

\[
\begin{align*}
\vec{E} &= \text{Eletric field vectors} \\
\vec{H} &= \text{Magnetic field vectors} \\
\vec{D} &= \text{Eletric displacement} \\
\vec{B} &= \text{Magnetic inductance}
\end{align*}
\]

\[
\begin{align*}
\vec{P} &= \text{polarization} \\
\vec{M} &= \text{magnetization} \\
\rho &= \text{charge density} \\
\vec{j} &= \sigma \vec{E} = \text{current density}
\end{align*}
\]
2.9 Maxwell’s Equations

- Let’s combine I and II (assume no free charge: $\rho = 0, \vec{j} = 0$)
- Use property: $\nabla \times (\nabla \times \vec{E}) = \nabla (\nabla \vec{E}) - \nabla^2 \vec{E}$
- Since $\rho = 0 \Rightarrow \nabla \vec{E} = 0 \Rightarrow \nabla \times (\nabla \times \vec{E}) = -\nabla^2 \vec{E}$
- Take $\nabla \times (\text{Eq} I)$:

$$\nabla \times (\nabla \times \vec{E}) = -\nabla \times \left( \frac{\partial \vec{B}}{\partial t} \right) \quad (2.46)$$

$$\Rightarrow -\nabla^2 \vec{E} = \frac{\partial}{\partial t} (\nabla \times \vec{B}) =$$

$$= -\frac{\partial}{\partial t} (\mu_o \nabla \times \vec{H}) =$$

$$= -\frac{\partial}{\partial t} (\mu_o \frac{\partial \Delta}{\partial t}) =$$

(see next slide)
2.9 Maxwell’s Equations

\[ -\nabla^2 \bar{E} = \frac{\partial}{\partial t} (\nabla \times \bar{B}) = \]

\[ = -\frac{\partial}{\partial t} (\mu_0 \nabla \times \bar{H}) = \]

\[ = -\frac{\partial}{\partial t} (\mu_0 \frac{\partial \bar{A}}{\partial t}) = \]

\[ = -\varepsilon \mu \frac{\partial^2 E}{\partial t^2} \]

Thus:

\[ \nabla^2 \bar{E} = -\varepsilon \mu \frac{\partial^2 E}{\partial t^2} = 0 \quad (2.47) \]

- Wave Equation
2.9 Maxwell’s Equations

Note:
\[
\begin{align*}
\varepsilon \mu &= \frac{1}{v^2} ; \quad v = \frac{c}{n} ; \quad c = \text{speed of light in vacuum} \\
n &= \sqrt{\frac{\mu \varepsilon}{\mu_0 \varepsilon_0}}
\end{align*}
\]

- The wave equation describes the propagation of a time-dependent field (eg. pulse)
- **Solution**: plane wave: \( E = E_0 e^{-i(\omega t - k \cdot r)} \)
  \[
  k = \frac{2\pi}{\lambda} = \frac{\omega}{v} = n \frac{\omega}{c} ; \quad k = \text{wave equation}
  \]
2.9 Maxwell’s Equations

- Phase of the field:
  \[ \varphi = \omega t - \vec{k} \cdot \vec{r} \]  
  \[ \text{(2.48)} \]

- **Note**: \( \varphi = \) constant describes a surface that moves with a certain velocity.

  \[ \omega t - \vec{k} \cdot \vec{r} = \text{constant} \]  
  eq of planes \( \perp \vec{k} \)

  \[ \Rightarrow \omega dt - kdr = 0 \]

  \[ \Rightarrow \frac{dr}{dt} = \frac{\omega}{k} = v_p \]  
  \[ \text{(2.49)} \]

  The surface of constant phase is traveling with velocity:

  \[ v_p = \frac{\omega}{k} = \text{phase velocity} \]
2.9 Maxwell’s Equations

- What is the counterpart of the wave equation for the frequency domain?
- Well, remember \( \frac{\partial}{\partial t} \rightarrow i\omega \)
- Upon Fourier transforming, Eq. 2.47 becomes:

\[
\nabla^2 \bar{E} - \frac{1}{v^2} (i\omega i\omega) E(\omega) = 0
\]

\[
\Rightarrow \nabla^2 \bar{E} + \frac{1}{v^2} (\omega^2) E(\omega) = 0
\]

- Note: \( k = \frac{\omega}{v} \)

\[
\Rightarrow \nabla^2 E(\omega) + k^2 E(\omega) = 0 \quad (2.50)
\]
2.9 Maxwell’s Equations

$$\nabla^2 E(\omega) + k^2 E(\omega) = 0$$

- The equation above is the “Helmholtz equation”
- Describes how each frequency $\omega$ propagates