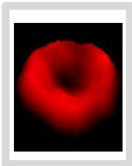

Optical Imaging
Chapter 5 – Light Scattering

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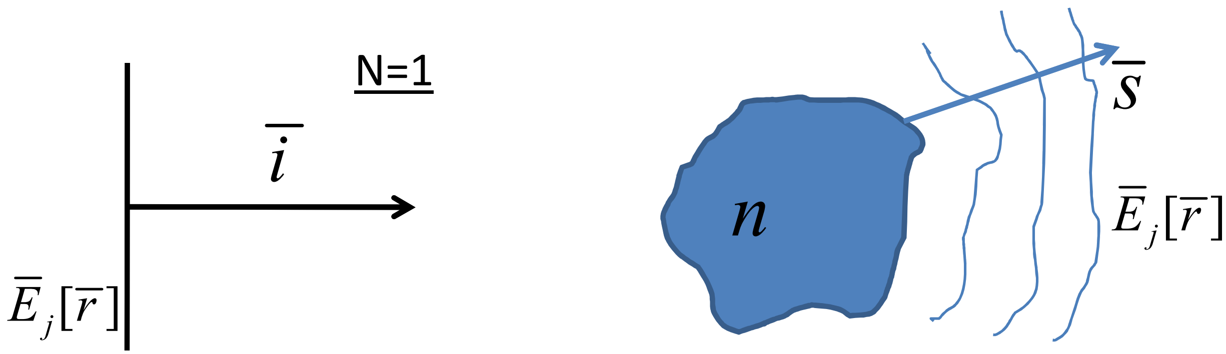
Light Scattering from Homogeneous Media

- "Scattering" is a generic term for the interaction of radiation by potentials.
- Neutron scattering on mass (nuclei)
- X-ray scattering on charge (ē)
- Scattering of an electromagnetic field on "scattering potentials" i.e. dielectric homogeneous

5.1 Scattering on Simple Particles



- Direct Problem: given $n[\bar{r}]$, what is $E[\bar{s}]$?
- Inverse Problem: given $E[\bar{s}]$, what is $n[\bar{r}]$?
- This problem is more difficult but it provides a diagnostics tool

$$\bar{E}_j[r] = \bar{E}_0 \cdot e^{i \cdot \bar{k} \cdot \bar{n}} = \text{plane move incident}; \quad \left|\bar{k}\right| = \frac{2\pi}{\lambda} \qquad (5.1)$$

5.1 Scattering on Simple Particles

- Particles can be characterized by a complex dielectric constant

$$\varepsilon[\bar{r}] = \mathrm{Re}[\varepsilon[\bar{r}]] + i\,\mathrm{Im}[\varepsilon[\bar{r}]] \quad (5.2)$$

$$\bar{\varepsilon}[\bar{r}] = n^2[\bar{r}] \quad (5.3)$$

- Recall:
- Re[n] ➔Refraction
- Im[n] ➔Absorption
- The scattered "far" field ($R > \frac{d^2}{\lambda}$)

$$\bar{E}_S[\bar{r}] = \bar{E}_0 \cdot \frac{e^{i \cdot k \cdot R}}{R} \cdot f[\bar{s}, \bar{i}] \quad (5.4)$$

$f[\bar{s}, \bar{i}] =$ scattering amplitude

= defines amplitude, phase, and polarization of the scattered field.

5.1 Scattering on Simple Particles

- Differential Cross section

$$\sigma_d[\bar{s},\bar{i}] = \lim_{R\to\infty}[R^2 \left|\frac{\bar{S}_s}{\bar{S}_i}\right|] = \left|f[\bar{s},\bar{i}]\right|^2 \quad (5.5)$$

- Only defined in far-field
- $\bar{S}_i$ and $\bar{S}_s$ are the Pointing vectors (incident and scattered.)

$$\begin{cases} S_{i,s} = \dfrac{1}{2\eta}\left|\bar{E}_{i,s}\right|^2 \\ \eta = \sqrt{\dfrac{\mu}{\varepsilon}} = \text{impedance } (\eta_{vacuum} = 377\Omega) \end{cases} \quad (5.6)$$

- From (eq. 5.5) differential cross-section:

$$\sigma_b = \sigma_d[-\bar{i},\bar{i}] \quad (5.7)$$
$$= \text{back scattering section}$$

5.1 Scattering on Simple Particles

- Scattering Cross section

$$\sigma_s = \int_{4\pi} \sigma_d[\bar{s}, \bar{i}]\, d\Omega \qquad (5.8)$$

$$[\sigma_s] = m^2 = \text{area} = \text{meaning of probability of scattering}$$

- Phase Function

$$p[\bar{s}, \bar{i}] = 4\pi \frac{\sigma_d[\bar{s}, \bar{i}]}{\sigma} = 4\pi \frac{\left| f[\bar{s}, \bar{i}] \right|^2}{\sigma} \qquad (5.9)$$

$$\int p[\bar{s}, \bar{i}]\, d\Omega = 4\pi = \text{normailzation}$$

5.1 Scattering on Simple Particles

- In the presence of absorption

$$\sigma = \sigma_a + \sigma_s = \text{Total Cross Section Area} \qquad (5.10)$$

- Most of the time we use "spherical" particles approximation.
- Various regimes, depending on $\frac{a}{\lambda}$ and (n-1): Rayleigh Scattering, Rayleigh-Gans, Mie
- Collection of particles: density is important
- Again regimes: single scattering, multiple scattering, and diffusion.

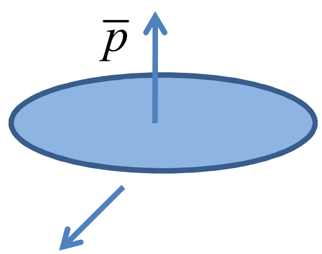
5.2 Rayleigh Scattering

- If $a << \lambda$, the dipole approximation works well
- Scattering Amplitude

$$\varepsilon = n^2$$

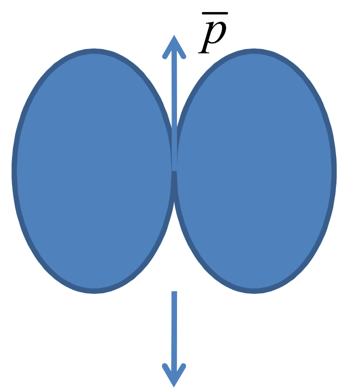
$$f[\bar{s},\bar{i}] = \frac{k^2}{4\pi}\left[\frac{3(\varepsilon_r - 1)}{\varepsilon_r + 2}\right] \cdot V \cdot [-\bar{s} \times (\bar{s} \times \bar{i})] \qquad (5.11)$$

- It describes the “doughnut” shape



isotropic in the plane $\perp\ \bar{p}$

$\bar{p}$ = induced dipole moment (think Lorentz)



no radiation along $\bar{p}$

5.2 Rayleigh Scattering

- Question: is the sky polarized?

$$\sigma_s = \pi a^2 \frac{8(ka)^4}{3} \cdot \left| \frac{(\varepsilon_r - 1)}{\varepsilon_r + 2} \right|^2 \tag{5.12}$$

- Note:

$$\sigma \sim k^4 = \frac{2\pi}{\lambda^4} \Rightarrow \text{ small } \lambda \text{ scatter more } \Rightarrow \text{ blue}$$

$$\sigma_s \sim a^6 = V^2$$

5.3 The Born Approximation

- Recall the Helmholtz equation (Chapter 2, eq. 2.50)

$$\nabla^2 U[\overline{r},\omega] + k^2 n^2[\overline{r},\omega] U[\overline{r},\omega] = 0$$

$$k = \frac{2\pi}{\lambda}; \qquad \varepsilon = n^2 \tag{5.13}$$

- Scalar eq. is good enough for our purpose
- The Scattering Potential

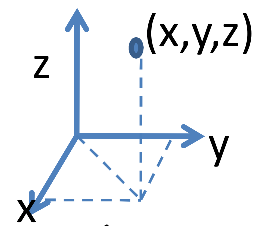
$$F[\overline{r},\omega] = \frac{1}{4\pi} k^2 \cdot (n^2[\overline{r},\omega] - 1) \tag{5.14}$$

=> Equation 5.13 can be rewritten:

$$\nabla^2 U[\overline{r},\omega] + k^2 U[\overline{r},\omega] = -4\pi F[\overline{r},\omega] \cdot U[\overline{r},\omega] \tag{5.15}$$

5.3 The Born Approximation

- Let's use the Fourier method for linear differential equations. Instead of (t,ω) domain, use space – spatial frequency domain, $(\overline{r},\overline{q})$. (5.12)
- Let's find the Green's function, i.e. impulse response in space domain. Space-Pulse = $\delta^{(3)}[\overline{r}]$



$\delta[x-x'].\delta[y-y'].\delta[z-z']$

- The elementary equation becomes:

$$\nabla^2 H[\overline{r},\omega]+k^2 H[\overline{r},\omega]=-\delta^{(3)}[\overline{r}] \quad (5.16)$$

- Take the Fourier Transform w.r.t. $\overline{r}$

$$\Rightarrow -q^2\cdot H[\overline{q},\omega]+k^2\cdot H[\overline{q},\omega]=-1 \quad (5.17)$$

5.3 The Born Approximation

- (Eq. 5.17 from last slide)

$$\Rightarrow -q^2 \cdot H[\overline{q},\omega] + k^2 \cdot H[\overline{q},\omega] = -1 \qquad (5.17)$$

$$\Rightarrow H[\overline{q},\omega] = \frac{1}{q^2 - k^2} \qquad (5.18)$$

- H = transfer function => impulse response.

$$\begin{cases} h[\overline{r},\omega] = F\{H[\overline{q},\omega]\} \\ \Rightarrow h[\overline{r},\omega] = A \cdot \dfrac{e^{ik\cdot|\overline{r}|}}{|\overline{r}|} \end{cases} \qquad (5.19)$$

- This is the <u>Spherical Wavelet</u> we used in diffraction (section 3.11)

5.3 The Born Approximation

- Scattering and diffraction are essentially the same thing : the interaction of light with a (usually small) object.
- So the solution of (eq. 5.15) is a convolution of the impulse response with $F[\bar{r},\omega]\cdot U[\bar{r},\omega]$:

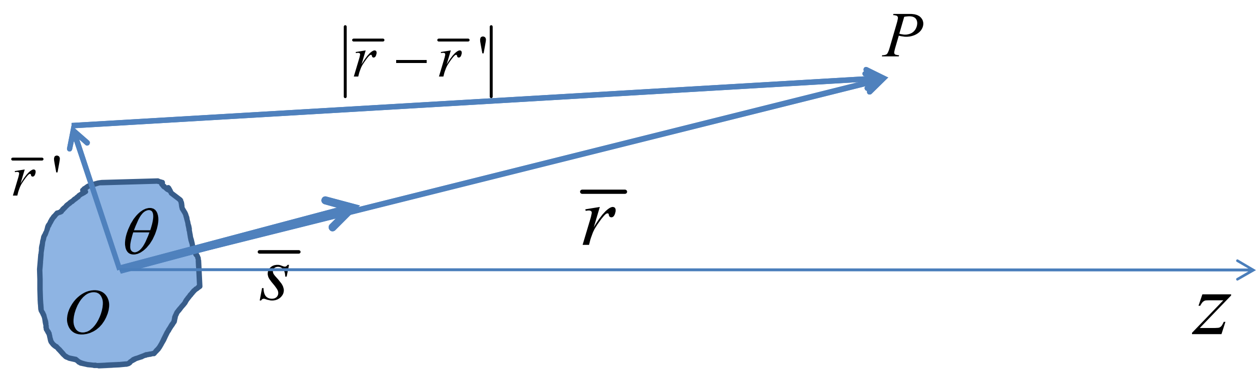
$$U^{(S)}[\bar{r},\omega]=\int_{(V)} F[\bar{r}',\omega]\cdot U[\bar{r}',\omega]\cdot\frac{e^{ik\cdot|\bar{r}-\bar{r}'|}}{|\bar{r}-\bar{r}'|}d^3r' \qquad (5.20)$$

$= \text{superposition of wavelets}$

- Assuming the plane wave incident $U^{(i)}=e^{ik\cdot\bar{s}_0\cdot\bar{r}}$, the total field in the for zone:

$$U[\bar{r},\omega]=e^{ik\cdot\bar{s}_0\cdot\bar{r}}+\overbrace{\int_{(V)} F[\bar{r}',\omega]\cdot U[\bar{r}',\omega]\cdot\frac{e^{ik\cdot|\bar{r}-\bar{r}'|}}{|\bar{r}-\bar{r}'|}d^3r'}^{U^{(s)}} \qquad (5.21)$$

5.3 The Born Approximation



- Useful Approximation (for zone)

$$|\bar{r}-\bar{r}\,'| = \sqrt{r^2 + r'^2 - 2rr'\cos\theta} = r\sqrt{1 + 2\frac{r'}{r}\cos\theta + \underbrace{\left(\frac{r'}{r}\right)^2}_{\simeq 0}}$$

- Since $r' << r$ and $\sqrt{1-2x} \simeq 1-x$

$$|\bar{r}-\bar{r}\,'| \simeq r - r'\cos\theta \qquad (5.22a)$$

- Equivalently $|\bar{r}-\bar{r}\,'| \simeq r - \bar{s}\cdot\bar{r}\,'$ (5.22b)

5.3 The Born Approximation

- Equation 5.22b implies that

$$\frac{e^{ik\cdot|\overline{r}-\overline{r}'|}}{|\overline{r}-\overline{r}'|} \simeq \frac{e^{ik\cdot r}}{r} e^{ik\cdot\overline{s}\cdot\overline{r}} \tag{5.23}$$

- The scattered field $U^{(S)}$ in equation 5.21 becomes

$$U^{(S)}[r\overline{s},\omega] = \frac{e^{ik\cdot r}}{r}\int F[\overline{r}',\omega]\cdot U[\overline{r}',\omega]\cdot e^{-ik\cdot\overline{s}\cdot\overline{r}'} d^3r' \tag{5.24}$$

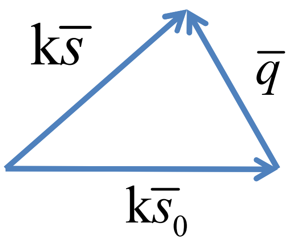
- Equation 5.24 is generally difficult to solve but a meaningful approximation can be made: the medium is <u>weakly scattering</u> = 1st Born Approximation.
- i.e. the field in the medium $\simeq$ incident field $\Rightarrow U[r',\omega] = e^{ik\cdot\overline{s}\cdot\overline{r}'}$

5.3 The Born Approximation

- With $U[r',\omega]=e^{ik\cdot\bar{s}\cdot\bar{r}'}$, equation 5.24 becomes the Born Approximation:

$$U^{(S)}[\bar{s},\bar{s}_0]=A\int F[r']\cdot e^{-ik\cdot(\bar{s}-\bar{s}_0)\cdot\bar{r}'}d^3r' \qquad (5.25)$$

- Scattering vector:
  $\bar{q}=k(\bar{s}-\bar{s}_0)$ → momentum transfer



- Note: $|\bar{p}|=|\bar{p}_0|$ → elastic scattering
- Equation 5.25 becomes:

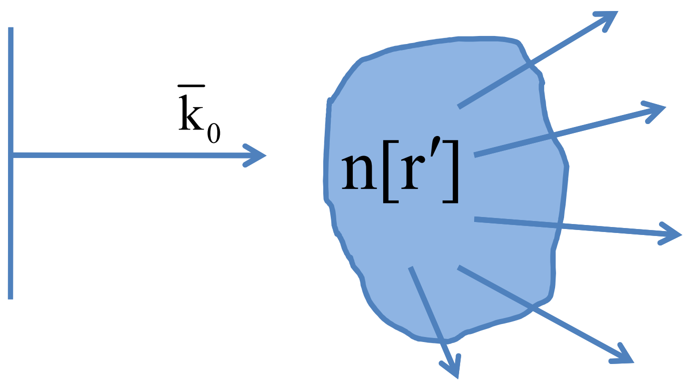
$$U^{(S)}[g]=A\int F[r']\cdot e^{-i\cdot\bar{q}\cdot\bar{r}'}d^3r' \qquad (5.26)$$

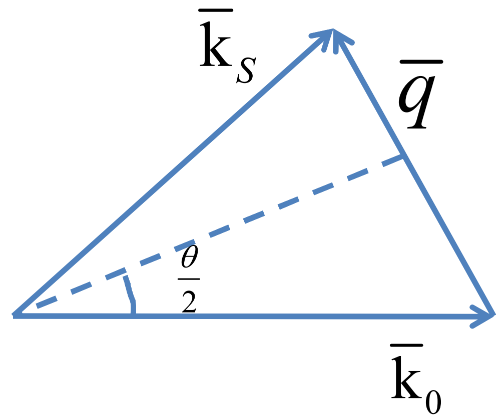
5.3 The Born Approximation

- Fourier Relationship:

$$\begin{cases} U^{(S)}[g] = \Im[F[\bar{r}']] \\ F[\bar{r}'] = \dfrac{1}{4\pi} k^2 \cdot (n^2[r'] - 1) \end{cases}$$

- F.T. is nice because it is easy to compute and <u>bi-directional</u>





$$|\bar{g}| = 2k_0 \sin\frac{\theta}{2}$$

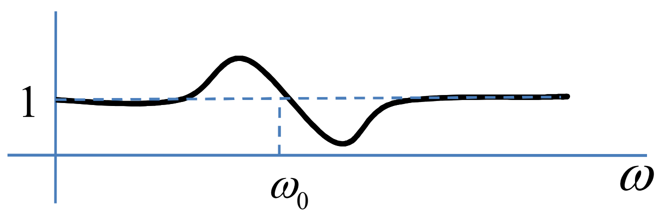
$\theta = scattering\ angle$

5.3 The Born Approximation

- So measuring $I[\theta]$ we can get info about $n[r]$ because

$$F[\overline{r}'] = \Im[U[\overline{q}]] \tag{5.27}$$

- Historically the Born approximation was first used in neutron scattering and then in x-rays
- Recall dispersion relationship n[ω]



$\lim_{\omega \to \infty} n[\omega] = 1$ → for x rays, most materials are within Born Approx

- Spectroscopy Lab uses light scattering to detect early cancer!!

5.4 The Spatial Correlation Function

- $U^{(s)}(q) = A\int F(r')e^{-i\bar{q}\cdot\bar{r}'}d^3r'$ provides the scattered field $U^{(s)}$
- The measured quantity is $\left|U^{(s)}\right|^2 = I^{(s)}$

  $$I^{(s)} = \left|U^{(s)}\right|^2 = \Im[F(r)]\Im[F(r')]^*$$

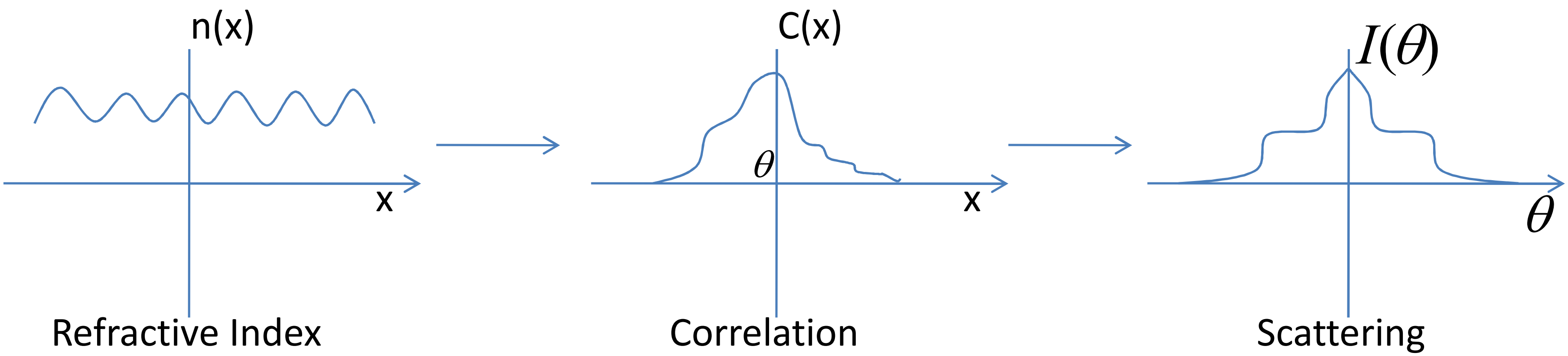
- Recall correlation theorem: $GG^* = \Im[g \otimes g]$
- Product of Fourier Transformation equals $\Im$ of correlation.
- $F(r') = \Im[U(\bar{g})]$ becomes:

  $I^{(s)} = \Im[F(r) \otimes F(r)]$ where $C(r) = \int f(r')f(r'-r)d^3r'$

  $$I^{(s)}(q) = \int C(r)e^{-i\bar{q}\cdot\bar{r}}d^3\bar{r} \quad (4.28)$$

5.4 The Spatial Correlation Function

- $I^{(s)}(\overline{q}) = \int C(\overline{r}) e^{-i\overline{g}\cdot\overline{r}} d^3\overline{r}$ is another form of the Born approximation
- It connects measurable quantity $I^{(s)}(g)$ with the spatial correlation function $C(\overline{r})$.
- $I^{(s)}(q) = \int C(\overline{r}) e^{-i\overline{g}\cdot\overline{r}} d^3\overline{r}$ applies to both continuous and discrete media.



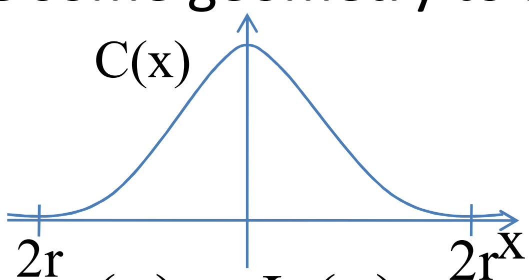
5.5 Single Particle Under Born Approximation

- Spherical particle $\left| \begin{array}{l} n(r) = n, \text{ if } |n| < R \\ 0, \text{ rest} \end{array} \right.$

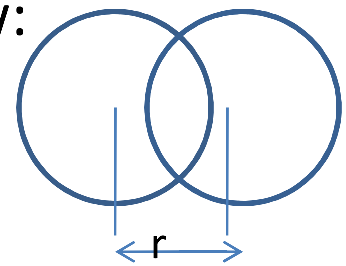
- Correlation means the volume of overlaps between shifted "versions" of the function:

$$C(r) = \int f(r')f(r'-r)d^3r' \quad (5.29)$$

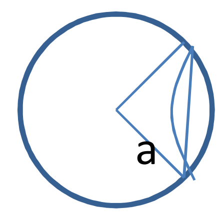
- Use some geometry to show:



$\sigma_d(q) = I_d(q)$



The overlap vol is 2x of thespherical "cap"



$$C(r) = A[1 - \frac{3r}{4R} + \frac{1}{16}\left(\frac{r}{R}\right)^3] \quad (5.30)$$

- The differential cross section of such particle is $\sigma_d(q) = I_d(q)$

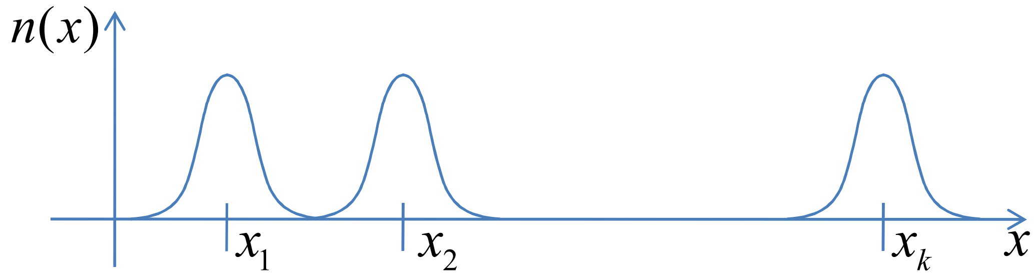
5.5 Single Particle Under Born Approximation

- Using: $I^{(s)}(q) = \int c(r) e^{-i\overline{q}\cdot\overline{r}} d^3\overline{r}$
- $$\sigma_d(q) = A\frac{1}{q^4 v^6}[\sin(qa) - ga\cos(qa)]^2 \qquad (5.31)$$
- $\sigma_d(q) = A\frac{1}{q^4 v^6}[\sin(qa) - ga\cos(qa)]^2$ is the solution for Rayleigh-Gauss particles (Born approx. for spheres).

- Applies for $kd(n-1) \ll 1$
- Greater applicability than Rayleigh(Rayleigh is a particular case of Rayleigh-Gaus, for a → 0)

5.6 Ensemble of particles

- Example:



$$n(x) = f(x) \circledast [\sum_i \delta(x - x_i)]$$ → Convolution (5.32)

- Particles have refractive index given by f(x), but are distributed in space at positions $\delta(x - x_i)$
- Consider volume distribution of identical particles:

$$F(r) = F_0(r) \circledast \sum d(\overline{r} - \overline{r}_i)$$ (5.33)

5.6 Ensemble of particles

- Particles have refractive index given by f(x), but are distributed in space at positions
- Consider volume distribution of identical particles:

$$F(r) = F_0(r) \circledV \sum d(\overline{r} - \overline{r}_i) \quad (5.33)$$

- $F_0(r) =$ scattering potential a single particle.
- $\circledV$ convolution
- The scattered field is (Eq. 5.26)

$$U^{(s)}(g) = \Im[F_0(\overline{r}) \circledV \sum_i d(\overline{r} - \overline{r}_i)] \quad (5.34)$$

$$= \underbrace{\Im[F_0(\overline{r})]}_{F(g)} \underbrace{\Im[\sum_i d(\overline{r} - \overline{r}_i)]}_{S(g)}$$

5.6 Ensemble of particles

- The scattered field is (Eq. 5.26)

$$U^{(s)}(g) = \Im[F_0(\overline{r}) \circledV \sum_i d(\overline{r} - \overline{r}_i)] \tag{5.34}$$

$$= \underbrace{\Im[F_0(\overline{r})]}_{F(g)} \underbrace{\Im[\sum_i d(\overline{r} - \overline{r}_i)]}_{S(g)}$$

$F(g)$ = Form factor

= Describe the scattering from one particle

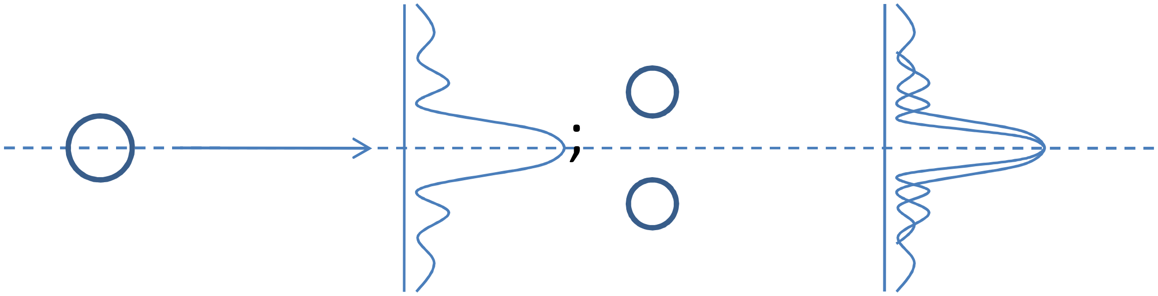
$S(g)$ = Structure factor – interference

= Defines the arrangement of scattering

5.6 Ensemble of particles

- Similarly

a) Yang experiment



b) X-ray scattering on crystals

- each scattering is Rayleigh are isotropic in angle (atom)
- Structure factor provides info about lattice

5.7 Mie Scattering

- Provides exact solution σ_s for spherical particles of arbitrary refractive index and size.

$$\sigma_s = \pi a^2 \frac{2}{\alpha^2} \sum_{n=1}^{\infty} (2n+1)\left(|a_n|^2 + |b_n|^2\right) \qquad (5.35)$$

with a_n, b_n = Mie coefficients = complicated functions of $\alpha = ka$

$$\beta = k\omega a$$

$$\omega = \frac{n}{n_0}$$

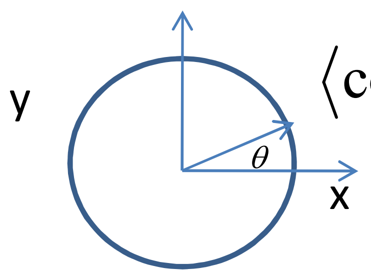
- The above equation is an infinite series
- Converges slower for larger particles
- Computes routines easily available

5.7 Mie Scattering

- Scattering Efficiency

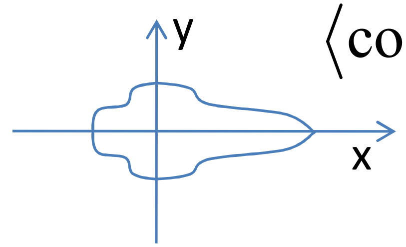
$$Q = \frac{\sigma_s}{\pi a^2} \qquad (5.36)$$

- Scattering from large particles:



$$\langle \cos(\theta) \rangle = 0$$

Rayleigh



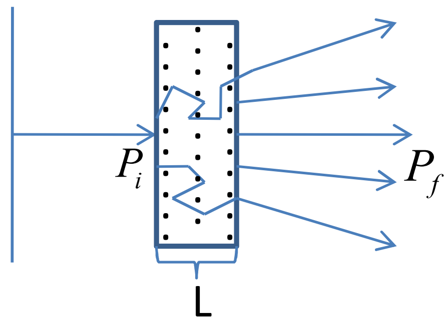
$$\langle \cos(\theta) \rangle = g > 0$$

g = anisotropy factor

Nie -Large Sphere

- Although Mie is restricted to spheres, is heavily used for tissue modeling
- Mie takes into account multiple bounces inside the particle.

5.8 Multiple Light Scattering



- Attenuation through scattering medias is quantified by the scattering mean free path:

$$l_s = \frac{1}{N\sigma_s} \qquad (5.37)$$

- N = Concentration of particles [m^{-3}]

- Scattering Coefficient: $\mu = \frac{1}{l_s} = N\sigma_s$

- The power left unscattered upon passing through the medium: $\boxed{P_f = P_i e^{-\mu_s L}}$ (5.38)

= Lambert-Beer

5.8 Multiple Light Scattering

- Transport mean free path:

$$l_t = \frac{l_s}{1-g} \qquad g = \langle \cos(\theta) \rangle \qquad (5.39)$$

= average of scatter angle

- l_t is renormalizing l_s ; takes into account the directional scattering of large particles.
- Note:

$$g \in (0.75; 0.95) \text{ for tissue!}$$

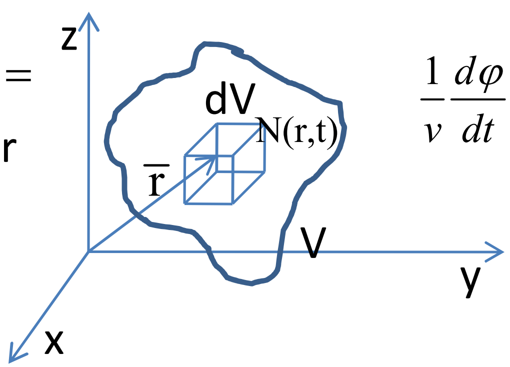
$$\frac{l_t}{l_s} \in (10; 20) \text{-important factor}$$

5.9 The Transport Equation

- The multiple scattering regime is not tractable for the general case.
- For high concentration of particles some useful approximations can be made.
- Photon random walk model → borrowed from nuclear reaction theory-> Transport Equation = Balance of Energy.

$\varphi(r,\Omega,t) =$

= angular flux



$$\frac{1}{v}\frac{d\varphi}{dt} + \bar{\Omega}\bar{\nabla}\varphi + \mu\varphi(\bar{r},\Omega,t) - \int_{4\pi}[\mu_s(\bar{\Omega}'\to\Omega)\varphi(\bar{r},\Omega',t)]d\Omega' = S(\bar{r},\bar{\Omega},t) \quad (5.40)$$

Leakage out of dV; Loss due to scott; Scattered into direction Ω; Source

- Even this simple equation is typically solved numerically.
- One more approximation → analytic solutions.

5.10 The Diffusion Approximation

- Integrate both sides of the equation $S(\overline{r},\overline{\Omega},t)$ with respect to solid angle Ω
- $$\frac{1}{v}\frac{d\phi}{dt}+\overline{\nabla}\overline{J}(\overline{r},t)+\mu\phi(\overline{r},t)=\mu_s\phi(\overline{r},t)+S(\overline{r},t) \qquad (5.41)$$

$$\phi = \text{ photon flux } = \int_{4\pi}\varphi(\overline{r},\overline{R},t)d\overline{\Omega} \qquad \overline{J} = \text{ current } = \int_{4\pi}\overline{\Omega}\varphi(\overline{r},\overline{R},t)d\overline{\Omega}$$

5.10 The Diffusion Approximation

- Diffusion Approximation:
- Assume the angular flux is only linearly anisotropic.

$$\varphi(\bar{r},\bar{\Omega},t) \simeq \frac{1}{4\pi}\phi(\bar{r},t) + \frac{3}{4\pi}\bar{J}(\bar{r},t)\cdot\bar{\Omega} \quad (5.42)$$

($\bar{J}(\bar{r},t)\cdot\bar{\Omega}$ → $\cos(\theta)$)

- If $\frac{1}{|\bar{J}|}\frac{\partial \bar{J}}{\partial t} << v\mu_t$ = slow variation of J, an equation that relates $\bar{J}$ and ϕ can be derived:

$$\bar{J}(\bar{r},t) = -\frac{1}{3\mu_t}\nabla\phi(\bar{r},t) = -D(\bar{r})\bar{\nabla}\phi(\bar{r},t) \quad (5.43)$$

$\mu t = \frac{1}{l_t}; l_t = \frac{l_s}{1-g}; g = \langle \Omega\cdot\Omega' \rangle$ = average cosine.

$D =$ Photon diffusion Coefficient

- Eq 5.43 is Flick's Law

5.10 The Diffusion Approximation

- The diffusion equation can be derived:

$$\frac{1}{v}\frac{d\phi}{dt} - \nabla[D(r)\nabla\phi(\overline{r},t)] + \mu_a(r)\phi(\overline{r},t) = S(\overline{r},t) \qquad (5.44)$$

- Equation in only one variable: $\phi = vr(\overline{r},t) =$ photon flux

  $v =$ photon velocity

  $n =$ photon density $[\mathrm{m}^{-3}]$

- μ_a = absorption coefficient
- For homogeneous medium and no absorption:

$$\frac{d\phi(\overline{r},t)}{dt} - vD\nabla^2\phi(\overline{r},t) = vS(r,t) \qquad (5.45)$$

→ Time resolved equation

5.10 The Diffusion Approximation

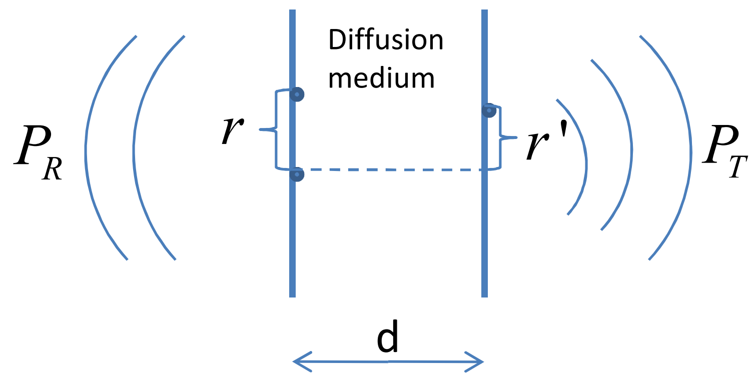
- Diffusion equation often used for light-tissue introduction
- Applicability: $L >> l_s \Leftrightarrow$ many scattering events

  $\mu_a << \mu_s \rightarrow$ absorption not too high

- ϕ = measurable quantity
- Diffusion model gives insight into the medium/tissue
- Measurable parameters: μ_s , μ_a , g , etc

5.11 Solutions of the Diffusion Equation

- Slab of thickness d:



- <u>Power reflected</u>:

$$P_R(r,s,d) = A \cdot s^{-\frac{5}{2}} e^{-\mu_a s} e^{-\frac{r^2}{4Ds}} f_R(d, z_e)$$

$$P_T(r,s,d) = A \cdot s^{-\frac{5}{2}} e^{-\mu_a s} e^{-\frac{r^2}{4Ds}} f_T(d, z_e) \quad (5.46)$$

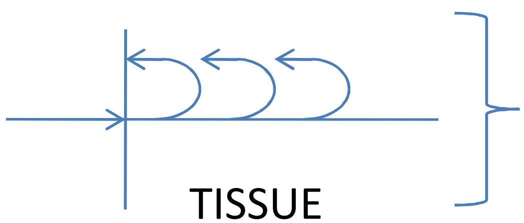
- With: s = path-length = v_t

  z_e = extrapolated length → boundary condition
- f_R, f_T reflection and transmission functions
- Laser pulse investigation has been used for tissue
- Stationary investigation also used → <u>spatially resolved</u>

5.12 Diffusion of Light in Tissue

- Tissue is a highly scattering medium; direction & polarization are randomized.
- Typical values: l_s=100μm, $g \simeq 0.9$
- Measurable quantities:
  - μ_a = absorption factor
  - g = anysotropy factor
  - μ_s = scattering coefficient
- Reflection and geometry is suitable for in vivo diagnostics

a) Time resolved:



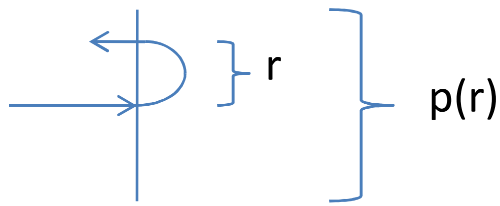
$P(s) =$ path-legth distribution

5.12 Diffusion of Light in Tissue

a) Time resolved:

- Used with femtosecond laser and LCI
- P(s) is a measurable tissue characterization

b) Spatially resolved:



- Note:
  - Semi-infinite medium model is heavily used.
  - If there is refractive index at boundary, not quite continuous angular distribution.