ECE 659 MIDTERM EXAM

March 9, 2009 930A-1020A

CLOSED BOOK (Notes provided)

NAME : SOLUTION

PUID# : _____

Please show all work and write your answers clearly.

This exam should have seven pages.

Problem 1	[p. 2]	6 points
Problem 2	[p. 3]	6 points
Problem 3	[p. 4]	6 points
Problem 4	[p. 5, 6]	6 points
Problem 3	[p. 7, 8]	6 points
Total		30 points

Problem 1: A two-dimensional conductor has a single band with an isotropic E(k) relation of the form $E = Ak^{\alpha}$, where A and α are constants. Starting from the expression for the conductivity

$$\sigma_{zz} = q^2 \int dE \left(-\frac{\partial f}{\partial E}\right) \frac{D(E)}{WL} \frac{v^2 \tau}{2}$$

show that at low temperatures (for which $-\partial f / \partial E$ can be approximated by a delta function at E $= \mu$), irrespective of A and α, we can write the sheet conductivity $\sigma_{zz} = q^2 n_s \tau / m$, where n_s is the electron density and m is defined as $\hbar k / v$ evaluated at E $= \mu$.

$$\sigma_{zz} = q^2 \frac{D}{WL} \frac{v^2 \tau}{2}$$

$$= q^2 \frac{d}{dE} n_s(E) \cdot \frac{v^2 \tau}{2}$$

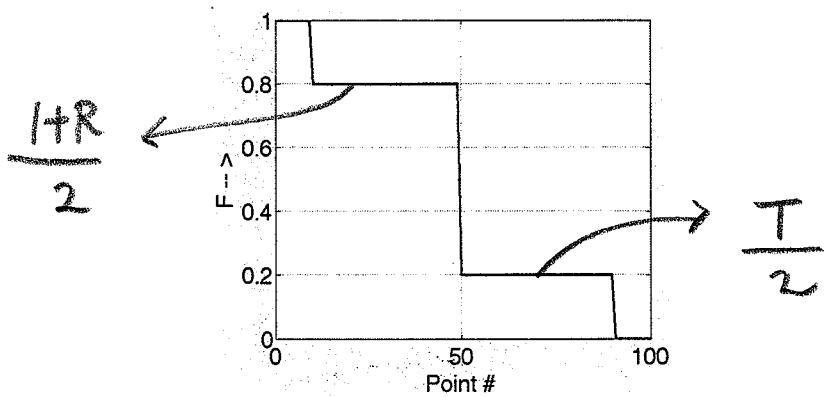
$$= q^2 \underbrace{\frac{dn_s}{dk}}_{\frac{d}{dk}\left(\frac{k^2}{4\pi}\right)} \cdot \frac{1}{\hbar v} \frac{v^2 \tau}{2}$$

$$= q^2 \cdot \frac{k}{2\pi\hbar} \frac{v\tau}{2}$$

$$= q^2 \frac{k^2}{4\pi} \frac{v\tau}{\hbar k} = q^2 n_s \tau * \frac{v}{\hbar k}$$

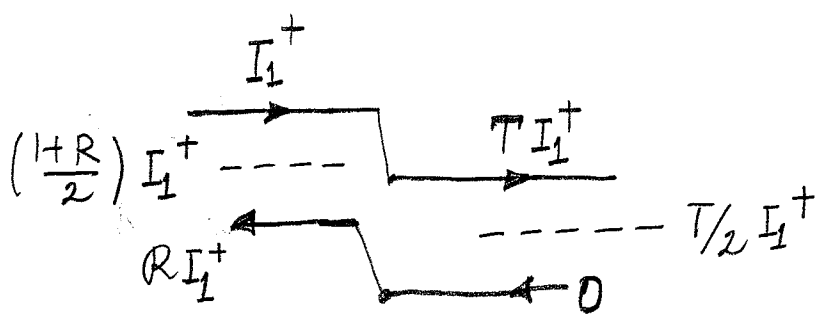
Problem 2: A conductor with M modes has one point scatterer having a scattering matrix of the form $\begin{Bmatrix} I_1^- \\ I_2^- \end{Bmatrix} = \begin{bmatrix} R & T \\ T & R \end{bmatrix} \begin{Bmatrix} I_1^+ \\ I_2^+ \end{Bmatrix} = \frac{q}{h} M \begin{bmatrix} R & T \\ T & R \end{bmatrix} \begin{Bmatrix} \mu_1^+ \\ \mu_2^+ \end{Bmatrix}$

with R+T = 1 ('+' denotes incoming flux, '-' denotes outgoing flux). Assuming $\mu_1^+ = qV$ and $\mu_2^+ = 0$, show that the average normalized electrochemical potential defined by $(\mu^+ + \mu^-)/2qV$ has the profile shown



and label the plateaus in the profile in terms of R and T.

What is the current $(I^+ - I^-)$?



$$I^+ - I^- = (1-R)\, I_1^+ = T I_1^+ = \frac{q^2 M T}{h} \cdot V$$

Problem 3: A 2-D conductor in a vector potential $\vec{A} = \hat{x}A_x + \hat{y}A_y$ is described by the Hamiltonian ($\vec{p} \equiv -i\hbar\vec{\nabla}$): $H_{op} = \frac{(p_x - qA_x)^2}{2m} + \frac{(p_y - qA_y)^2}{2m}$

Assume $A_x = -By$ and $A_y = 0$ (so that $\vec{\nabla} x \vec{A} = \vec{B}$).

Which of the following

A. $\psi = \exp(+ik_x x)\exp(+ik_y y)$ **B.** $\psi = \exp(+ik_x x)\,F(y)$ **C.** $\psi = F(x)\exp(+ik_y y)$

are acceptable solutions of the differential equation $E\psi = H_{op}\psi$ if

(a) B = 0

A, B, C

(b) B ≠ 0 ?

B only

Explain your reasoning briefly.

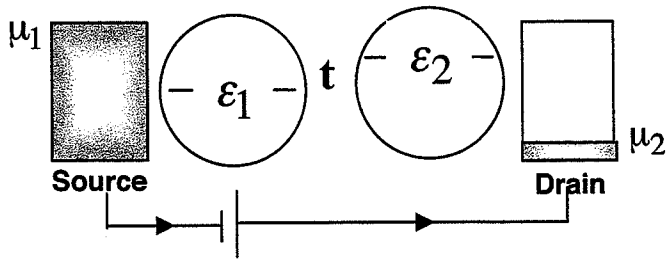
exp (+ikx) is an acceptable solution if coefficients do not vary in x.

same with y.

With B = 0, coefficients do not vary with x or y.

With B ≠ 0, coefficients vary with y, but not with x.

Problem 4:



Consider a device described by a (2x2) Hamiltonian $[H] = \begin{bmatrix} \varepsilon_1 & t \\ t^* & \varepsilon_2 \end{bmatrix}$ whose coupling to the contacts is described by $[\Sigma_1] = -\frac{i}{2}\begin{bmatrix} \gamma_1 & 0 \\ 0 & 0 \end{bmatrix}$, $[\Sigma_2] = -\frac{i}{2}\begin{bmatrix} 0 & 0 \\ 0 & \gamma_2 \end{bmatrix}$

Show that for small 't', the local density of states (LDOS) at points "1" and "2" can be written as $D_{1,2}(E) \approx \gamma_{1,2}\left|g_{1,2}\right|^2/2\pi$, where $g_{1,2}(E) \equiv 1/(E - \varepsilon_{1,2} + i\gamma_{1,2}/2)$.

Hint: The LDOS are related to the diagonal elements of the spectral function, [A(E)].

Also, you may find the following approximation helpful: $\begin{bmatrix} a & -t \\ -t^* & b \end{bmatrix}^{-1} \approx \begin{bmatrix} a^{-1} & a^{-1}tb^{-1} \\ b^{-1}t^*a^{-1} & b^{-1} \end{bmatrix}$

$$G = \begin{pmatrix} E - \varepsilon_1 + \frac{i\gamma_1}{2} & -t \\ -t^* & E - \varepsilon_2 + \frac{i\gamma_2}{2} \end{pmatrix}^{-1}$$

$$g_1 \equiv 1 \Big/ E - \varepsilon_1 + \frac{i\gamma_1}{2}$$

$$g_2 \equiv 1 \Big/ E - \varepsilon_2 + \frac{i\gamma_2}{2}$$

$$\simeq \begin{pmatrix} g_1 & g_1 t g_2 \\ g_2 t^* g_1 & g_2 \end{pmatrix}$$

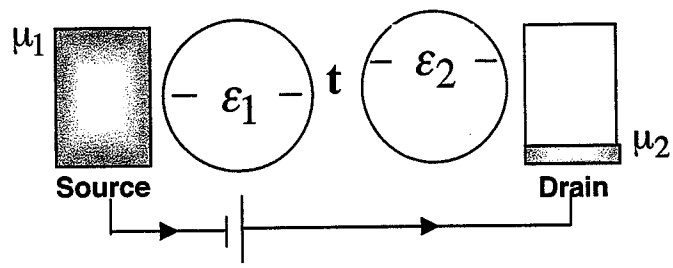
$$D_1 = i\left(g_1 - g_1^*\right)/2\pi$$

$$= i\left(\frac{1}{E-\varepsilon_1+i\gamma_1/2} - \frac{1}{E-\varepsilon_1-i\frac{\gamma_1}{2}}\right)/2\pi$$

$$= \frac{\gamma_1/2\pi}{(E-\varepsilon_1)^2+(\gamma_1/2)^2} = \gamma_1 g_1 g_1^*/2\pi$$

$$D_2 = \gamma_2 g_2 g_2^*/2\pi$$

Problem 5:



Consider the same device as in Problem 4, described by a (2x2) Hamiltonian $[H]=\begin{bmatrix}\varepsilon_1 & t\\ t^* & \varepsilon_2\end{bmatrix}$ whose coupling to the contacts is described by

$$[\Sigma_1]=-\frac{i}{2}\begin{bmatrix}\gamma_1 & 0\\ 0 & 0\end{bmatrix}\ ,\ [\Sigma_2]=-\frac{i}{2}\begin{bmatrix}0 & 0\\ 0 & \gamma_2\end{bmatrix}$$

Show that for small 't', the transmission $\overline{T}(E)\equiv Trace[\Gamma_1 G\Gamma_2 G^+]$ can be written approximately as $4\pi^2 D_1(E)D_2(E)|t|^2$, where D_1(E) and D_2(E) are the local density of states at sites 1 and 2, which were shown in Problem 4 to be given by

$$D_{1,2}(E)\approx\gamma_{1,2}|g_{1,2}|^2/2\pi\text{, where } g_{1,2}(E)\equiv 1/(E-\varepsilon_{1,2}+i\gamma_{1,2}/2).$$

As in Problem 4, you may find the following approximation helpful:

$$\begin{bmatrix}a & -t\\ -t^* & b\end{bmatrix}^{-1}\approx\begin{bmatrix}a^{-1} & a^{-1}tb^{-1}\\ b^{-1}t^*a^{-1} & b^{-1}\end{bmatrix}$$

$$\overline{T}=Trace\underbrace{\begin{bmatrix}\gamma_1 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}g_1 & g_1tg_2\\ g_2t^*g_1 & g_2\end{bmatrix}}_{\begin{bmatrix}\gamma_1 g_1 & \gamma_1 g_1 t g_2\\ 0 & 0\end{bmatrix}}\underbrace{\begin{bmatrix}0 & 0\\ 0 & \gamma_2\end{bmatrix}\begin{bmatrix}g_1^* & g_2^*tg_1^*\\ g_1^*t^*g_2^* & g_2^*\end{bmatrix}}_{\begin{bmatrix}0 & 0\\ \gamma_2 g_1^*t^*g_2^* & \gamma_2 g_2^*\end{bmatrix}}$$

$$=\gamma_1\gamma_2|g_1|^2|t|^2|g_2|^2$$

$$=4\pi^2 D_1 D_2|t|^2$$