Problem 1

$$i\hbar \frac{d\psi}{dt} = \mu_B \left[\vec{\sigma} \cdot \vec{B} \right] \psi$$

$$i\hbar \frac{d\psi^{+}}{dt} = -\mu_B \, \psi^{+} \left[\vec{\sigma} \cdot \vec{B} \right]$$

$$\frac{d\vec{S}}{dt} = \frac{d\psi^{+}}{dt} \vec{\sigma} \psi + \psi^{+} \vec{\sigma} \frac{d\psi}{dt}$$

$$\frac{dS_m}{dt} = \frac{i\mu_B}{\hbar} \left(\psi^{+} (\vec{\sigma} \cdot \vec{B}) \sigma_m \psi - \psi^{+} \sigma_m (\vec{\sigma} \cdot \vec{B}) \psi \right)$$

$$= \frac{i\mu_B}{\hbar} \sum_n \left(\psi^{+} \underbrace{\sigma_n \sigma_m}_{i\,\epsilon_{nmp} \sigma_p} \psi - \psi^{+} \sigma_m \sigma_n \psi \right) B_n$$

$$= \frac{2\mu_B}{\hbar} \sum_{n,p} \psi^{+} \epsilon_{mnp} \sigma_p \psi \, B_n$$

$$= \frac{2\mu_B}{\hbar} \left(\vec{B} \times \vec{S} \right)_m$$

Problem 2,

$$H = \varepsilon_0 I + \varepsilon \sigma_z$$

$$\Sigma_1 = -\frac{i\alpha}{2}[I+\sigma_x], \quad \Gamma_1 = \alpha[I+\sigma_x]$$

$$\Sigma_2 = -\frac{i\alpha}{2}[I-\sigma_x], \quad \Gamma_2 = \alpha[I-\sigma_x]$$

$$G = [EI - H - \Sigma_1 - \Sigma_2]^{-1}$$

$$= [EI - \varepsilon_0 I - \varepsilon\sigma_z + i\alpha I]^{-1}$$

$$= \begin{pmatrix} E-\varepsilon_0-\varepsilon+i\alpha & 0 \\ 0 & E-\varepsilon_0+\varepsilon+i\alpha \end{pmatrix}^{-1}$$

$$= g_0 I + g_z \sigma_z$$

$$g_0 = \frac{(E-\varepsilon_0-\varepsilon+i\alpha)^{-1} + (E-\varepsilon_0+\varepsilon+i\alpha)^{-1}}{2}$$

$$g_z = \frac{(E-\varepsilon_0-\varepsilon+i\alpha)^{-1} - (E-\varepsilon_0+\varepsilon+i\alpha)^{-1}}{2}$$

$$A = a_0 I + a_z \sigma_z$$

$$a_0 = \frac{\alpha}{(E-\varepsilon_0-\varepsilon)^2+\alpha^2} + \frac{\alpha}{(E-\varepsilon_0+\varepsilon)^2+\alpha^2}$$

$$a_z = \quad '' \quad - \quad ''$$

$$G^{n} = G\Gamma_1 G^{+}$$

$$= (g_0 I + g_z \sigma_z)\, \alpha\, [I + \sigma_x]\, (g_0^* I + g_z^* \sigma_z)$$

$$= \alpha (g_0 I + g_z \sigma_z)(g_0^* I + g_z^* \sigma_z + g_0^* \sigma_x - i g_z^* \sigma_y)$$

$$= \alpha \Big[(g_0 g_0^* + g_z g_z^*) I + (g_0 g_0^* - g_z g_z^*) \sigma_x$$

$$+ (g_z g_0^* + g_0 g_z^*) \sigma_z + i (g_z g_0^* - g_0 g_z^*) \sigma_y \Big]$$

$$N = 2\alpha (g_0 g_0^* + g_z g_z^*)$$

$$S_x = 2\alpha (g_0 g_0^* - g_z g_z^*)$$

$$S_y = 2i\alpha (g_z g_0^* - g_0 g_z^*)$$

$$S_z = 2\alpha (g_z g_0^* + g_0 g_z^*)$$

Problem 3

$$E\begin{pmatrix}\psi\\ \overline{\psi}\end{pmatrix} = \left(\frac{\hbar^2k^2}{2m}I + \hbar c E_0(\sigma_y k_x - \sigma_x k_y)\right)\begin{pmatrix}\psi\\ \overline{\psi}\end{pmatrix}$$

$$\begin{pmatrix} E - \frac{\hbar^2k^2}{2m} & -\hbar c E_0(-k_y - i k_x) \\ -\hbar c E_0(-k_y + i k_x) & E - \frac{\hbar^2k^2}{2m} \end{pmatrix}\begin{pmatrix}\psi\\ \overline{\psi}\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix}$$

$$\left(E - \frac{\hbar^2k^2}{2m}\right)^2 = (\hbar c E_0)^2 (k_x^2 + k_y^2)$$

$$E = \frac{\hbar^2k^2}{2m} \pm \hbar c E_0 k$$

Problem 4

(a)

$$\int_{-\infty}^{\infty} dE \left\{ D(+\varepsilon)\, G^n(E-\varepsilon)\left(A(E) - G^n(E)\right) - D(+\varepsilon)\, G^p(E+\varepsilon)\, G^n(E) \right\}$$

$$= D(+\varepsilon)\int dE\, G^n(E-\varepsilon)\, G^p(E) - D(+\varepsilon)\int dE\, G^n(E)\, G^p(E+\varepsilon)$$

here

$$\int_{-\infty}^{\infty} dE\, G^n(E-\varepsilon)\, G^p(E) = \int_{-\infty}^{\infty} dE'\, G^n(E')\, G^p(E'+\varepsilon)$$

hence, proved.

(b) In equilibrium,

$$D(+\varepsilon)\int dE\, G^n(E-\varepsilon)\, G^p(E) - G^n(E)\, G^p(E+\varepsilon)$$

$$+ D(-\varepsilon)\int dE\, G^n(E+\varepsilon)\, G^p(E) - G^n(E)\, G^p(E-\varepsilon)$$

$$= \int dE \left\{ \overbrace{D(+\varepsilon)\, G^n(E-\varepsilon)\, G^p(E) - D(-\varepsilon)\, G^n(E)\, G^p(E-\varepsilon)}^{1} + \underbrace{D(-\varepsilon)\, G^n(E+\varepsilon)\, G^p(E) - D(+\varepsilon)\, G^n(E)\, G^p(E+\varepsilon)}_{2} \right\}$$

For 1, $\dfrac{D(+\varepsilon)}{D(-\varepsilon)}\ \dfrac{G^n(E-\varepsilon)}{G^p(E-\varepsilon)}\ \dfrac{G^p(E)}{G^n(E)} = e^{-\varepsilon/kT}\, e^{-(E-\varepsilon-\mu)/kT}\, e^{(E-\mu)/kT}$

$= 1$ (in equilibrium)

hence 1 $= 0$, likewise 2 $= 0$.

Problem 5.

ECE 495N HW #9 Problem 3.

(http://sites.google.com/site/ece495n)

Problem 6.

ECE 495N HW # 4 Problem 4.

Problem 7.

$$P_n = \frac{1}{Z}\, {}^{N}C_n\, e^{-(n\varepsilon + U_0 + \alpha n - n\mu)/k_BT}$$

$$Z = \sum_n P_n = \sum_n {}^{N}C_n\, x^n\, e^{-U_0/k_BT} = e^{-U_0/k_BT}(1+x)^N$$

$$(x \equiv e^{-(\varepsilon + \alpha - \mu)/k_BT})$$

$$\langle n \rangle = \sum_n n P_n = \sum_n n\, {}^{N}C_n\, x^n\, e^{-U_0/k_BT} / Z$$

$$\sum_n {}^{N}C_n\, x^n = (1+x)^N, \quad \sum_n {}^{N}C_n\, n\, x^{n-1} = N(1+x)^{N-1}$$

$$\langle n \rangle = \frac{Nx(1+x)^{N-1}}{(1+x)^N} = \frac{Nx}{1+x}$$

$$f \equiv \frac{\langle n \rangle}{N} = \frac{1}{1+x^{-1}} = \frac{1}{1+e^{(\varepsilon+\alpha-\mu)/k_BT}}$$

For interacting electrons, one can use Fermi function assuming that electrons feel an average potential of $\alpha \equiv \frac{dU}{dn}$. This is the self-consistent potential one should use in a one-electron picture.

Problem 8

$$E_i - \mu N_i$$

$N = 0$: 0

$\mu = \varepsilon - t$

$N = 1$: $\varepsilon - t - \mu$

$\mu = \varepsilon + \frac{U}{2} + t - \frac{\sqrt{U^2+16t^2}}{2}$

$N = 2$: $\frac{2\varepsilon + U + 2\varepsilon - \sqrt{U^2 + 16t^2}}{2} - 2\mu$

$= 2\varepsilon + \frac{U}{2} - \sqrt{U^2+16t^2}/2 - 2\mu$

$\mu = \varepsilon + \frac{U}{2} - t + \frac{\sqrt{U^2+16t^2}}{2}$

$N = 3$: $3\varepsilon + U - t - 3\mu$

$\mu = \varepsilon + U + t$

$N = 4$: $4\varepsilon + 2U - 4\mu$