

**ECE 659      FINAL EXAM****May 7, 2009    LAWSON, B155    7PM - 9PM****CLOSED BOOK (4 pages of notes provided)**

NAME : \_\_\_\_\_

PUID# : \_\_\_\_\_

Please show all work and write your answers clearly.

This exam should have eleven pages.

Problem 1	[p. 2, 3]	6 points
Problem 2	[p. 4, 5]	6 points
Problem 3	[p. 6, 7]	6 points
Problem 4	[p. 8, 9]	6 points
Problem 5	[p. 10, 11]	6 points
Total		<hr/> 30 points

**Problem 1:** Consider a junction between two conductors having  $M_1$  and  $M_2$  modes respectively ( $M_1 > M_2$ ) described by a scattering matrix of the form

$$\begin{Bmatrix} I_1^- \\ I_2^- \end{Bmatrix} = \begin{bmatrix} (M_1 - M_2)/M_1 & 1 \\ M_2/M_1 & 0 \end{bmatrix} \begin{Bmatrix} I_1^+ \\ I_2^+ \end{Bmatrix} = \frac{q}{h} \begin{bmatrix} (M_1 - M_2) & M_2 \\ M_2 & 0 \end{bmatrix} \begin{Bmatrix} \mu_1^+ \\ \mu_2^+ \end{Bmatrix}$$

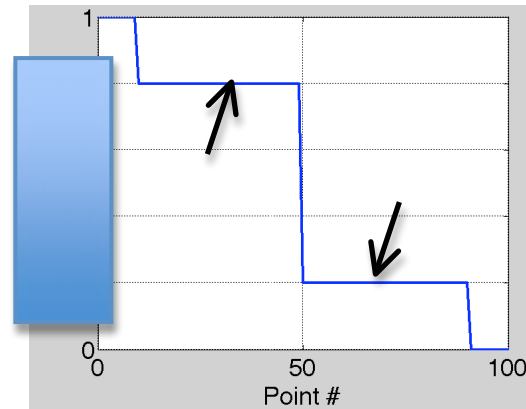
Assuming  $\mu_1^+ = qV$  and  $\mu_2^+ = 0$ ,

(a) The average normalized electrochemical potential defined by  $(\mu^+ + \mu^-)/2qV$  has the profile shown to the right :

Calculate and label the plateaus on either side of the junction marked by arrows.

(b) Calculate the current.

(c) Calculate the three resistances associated with each of the three drops in the potential.





**Problem 2:** A 2-D conductor is described by a Hamiltonian of the form

$$H = \frac{(p_x - qA_x)^2}{2m} + \frac{(p_y - qA_y)^2}{2m}$$

where the vector potential  $\vec{A} = \hat{x} A_x + \hat{y} A_y$  is constant in space.

(a) Find the dispersion relation  $E(k_x, k_y)$ .

(b) We wish to model it with a discrete square lattice having a nearest neighbor Hamiltonian whose non-zero elements are

$$H_{n,m} = \varepsilon \quad \text{if } \vec{d}_m = \vec{d}_n$$

$$H_{n,m} = -t_x \exp(-i\phi_x) \quad \text{if } \vec{d}_m - \vec{d}_n = \hat{x}a$$

$$H_{n,m} = -t_x \exp(+i\phi_x) \quad \text{if } \vec{d}_m - \vec{d}_n = -\hat{x}a$$

$$H_{n,m} = -t_y \exp(-i\phi_y) \quad \text{if } \vec{d}_m - \vec{d}_n = \hat{y}a$$

$$H_{n,m} = -t_y \exp(+i\phi_y) \quad \text{if } \vec{d}_m - \vec{d}_n = -\hat{y}a$$

All other elements are zero. How would you choose  $\varepsilon, t_x, \phi_x, t_y$  and  $\phi_y$  so as to obtain a good match to the dispersion relation for small energy E ?



**Problem 3:** Consider a channel with two spin levels, described by a (2x2) Hamiltonian

$$[H] = \varepsilon_0 I + \varepsilon \sigma_x$$

connected to two anti-parallel contacts 1 and 2, one communicating exclusively with +x spins and the other with -x spins:  $\Sigma_1 = (-i\alpha/2)[I + \sigma_x]$  and  $\Sigma_2 = (-i\alpha/2)[I - \sigma_x]$ . A small bias is applied so that over a small energy range around  $E = 0$  where we can assume  $f_1=1$  and  $f_2=0$ .

(a) Find the correlation function  $G^n$ .

(b) Find the number of electrons  $N$  and the net spin  $\vec{S}$  per unit energy in the channel.



**Problem 4:** Consider a device (temperature  $T$ ) with two sharp energy levels  $\varepsilon_1 - \varepsilon_2 = \hbar\omega$ , with the upper one in equilibrium with contact 1 and the lower one in equilibrium with contact 2. The current through this device can be written as

$$I = I_0 [F(-\hbar\omega) f_1(\varepsilon_1)(1 - f_2(\varepsilon_2)) - F(+\hbar\omega) f_2(\varepsilon_2)(1 - f_1(\varepsilon_1))]$$

where the absorption and emission rates characteristic of the surroundings (at temperature  $T_s$ ) are

given by 
$$F(+\hbar\omega) = \frac{1}{\exp(\hbar\omega/kT_s) - 1} \quad \text{and} \quad F(-\hbar\omega) = \exp(\hbar\omega/kT_s) F(+\hbar\omega)$$

and the contact Fermi functions are given by  $f_1(E) = f_0(E - \mu_1)$  and  $f_2(E) = f_0(E - \mu_2)$

where 
$$f_0(E) \equiv \frac{1}{\exp(E/kT) + 1}, \quad \mu_1 = E_f + (qV/2), \quad \mu_2 = E_f - (qV/2)$$

(a) Show that the current can be written in the form:

$$I = I_0 F(-\hbar\omega) (1 - f_0(\varepsilon_1 - \mu_1)) f_0(\varepsilon_2 - \mu_2) g(qV, \hbar\omega, kT, kT_s)$$

What is the function  $g(qV, \hbar\omega, kT, kT_s)$  ?

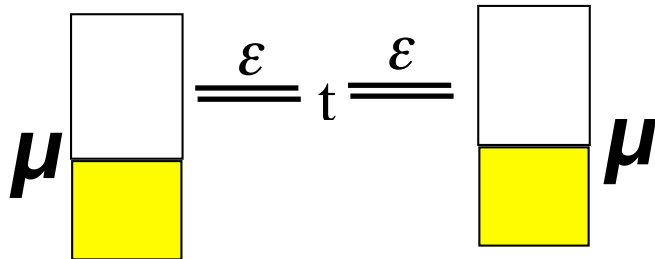
(b) What is the open circuit voltage in terms of  $\hbar\omega, kT, kT_s$  ?





**Problem 5:** Consider two coupled quantum dots each with two spin-degenerate levels, such that the one-electron Hamiltonian,  $h$

$$h = \begin{array}{c} \begin{array}{cccc} a & b & \bar{a} & \bar{b} \\ \varepsilon & t & 0 & 0 \\ t & \varepsilon & 0 & 0 \\ 0 & 0 & \varepsilon & t \\ 0 & 0 & t & \varepsilon \end{array} \end{array}$$



Dot 'a' has an extremely high interaction energy so that the state  $a\bar{a}$  is inaccessible in our energy range of interest and can be ignored. Only five (instead of the usual six) two-electron states need to be considered. Dot 'b' has zero interaction energy so that all five states  $b\bar{b}, a\bar{b}, b\bar{a}, ab, \bar{a}\bar{b}$  have the same diagonal element  $2\varepsilon$ .

What is the value of  $\mu$  (at low temperatures) at which the number of electrons inside the coupled quantum dot system will change from  $N = 1$  to  $N = 2$ ?

**Useful result:** Eigenvalues of  $\begin{bmatrix} p & q & q \\ q & p & 0 \\ q & 0 & p \end{bmatrix}$  are  $p, p \pm \sqrt{2}q$



*Have a great summer !*