ECE 659 FINAL EXAM

May 7, 2009 LAWSON, B155 7PM - 9PM

CLOSED BOOK (4 pages of notes provided)

NAME : SOLUTION

PUID# :

Please show all work and write your answers clearly.

This exam should have eleven pages.

Problem 1	[p. 2, 3]	6 points
Problem 2	[p. 4, 5]	6 points
Problem 3	[p. 6, 7]	6 points
Problem 4	[p. 8, 9]	6 points
Problem 5	[p. 10, 11]	6 points
Total		30 points

Problem 1: Consider a junction between two conductors having M_1 and M_2 modes respectively ($M_1 > M_2$) described by a scattering matrix of the form

$$\begin{Bmatrix} I_1^- \\ I_2^- \end{Bmatrix} = \begin{bmatrix} (M_1 - M_2)/M_1 & 1 \\ M_2/M_1 & 0 \end{bmatrix} \begin{Bmatrix} I_1^+ \\ I_2^+ \end{Bmatrix} = \frac{q}{h} \begin{bmatrix} (M_1 - M_2) & M_2 \\ M_2 & 0 \end{bmatrix} \begin{Bmatrix} \mu_1^+ \\ \mu_2^+ \end{Bmatrix}$$

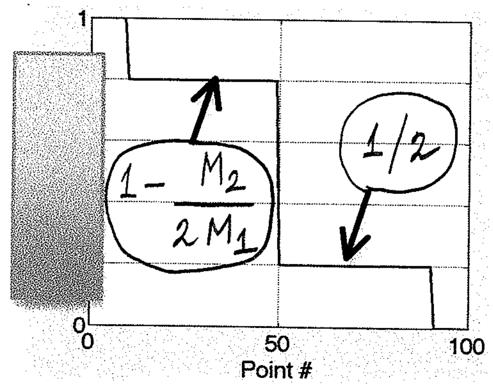
Assuming $\mu_1^+ = qV$ and $\mu_2^+ = 0$,

(a) The average normalized electrochemical potential defined by $(\mu^+ + \mu^-)/2qV$ has the profile shown to the right :

Calculate and label the plateaus on either side of the junction marked by arrows.

(b) Calculate the current.

(c) Calculate the three resistances associated with each of the three drops in the potential.



$$I_1^+ = \left(\frac{q}{h}\right) M_1 \mu_1^+ = \left(\frac{q^2 V}{h}\right) M_1 \, , \quad I_2^+ = 0$$

$$I_1^- = \left(\frac{q^2 V}{h}\right)(M_1 - M_2) \rightarrow \mu_1^- = (qV)\frac{M_1 - M_2}{M_1}$$

$$I_2^- = \left(\frac{q^2 V}{h}\right) M_2 \Rightarrow \mu_2^- = qV$$

$$(\mu_1)_{avg} = \frac{\mu_1^+ + \mu_1^-}{2qV} = \frac{1}{2} + \frac{M_1 - M_2}{2M_1} = 1 - \frac{M_2}{2M_1}$$

$$(\mu_2)_{avg} = \frac{\mu_2^+ + \mu_2^-}{2qV} = \frac{1}{2}$$

$$I = I_2^- = I_1^+ - I_1^- = \left(\frac{q^2 V}{h}\right) M_2$$

$$R_1 = \frac{V \cdot M_2/2M_1}{(q^2V/h)\, M_2} = \frac{h}{q^2} * \frac{1}{2M_1}$$

$$R_2 = \frac{V \cdot \left(\frac{1}{2} - M_2/2M_1\right)}{(q^2V/h)\, M_2} = \frac{h}{q^2} * \left(\frac{1}{2M_2} - \frac{1}{2M_1}\right)$$

$$R_3 = \frac{V \cdot 1/2}{(q^2V/h)\, M_2} = \frac{h}{q^2} * \frac{1}{2M_2}$$

$$R_1 + R_2 + R_3 = 1/M_2$$

Problem 2: A 2-D conductor is described by a Hamiltonian of the form

$$H = \frac{(p_x - qA_x)^2}{2m} + \frac{(p_y - qA_y)^2}{2m}$$

where the vector potential $\vec{A} = \hat{x}\,A_x + \hat{y}\,A_y$ is constant in space.

(a) Find the dispersion relation $E(k_x, k_y)$.

(b) We wish to model it with a discrete square lattice having a nearest neighbor Hamiltonian whose non-zero elements are

$$H_{n,m} = \varepsilon \quad if\ \vec{d}_m = \vec{d}_n$$

$$H_{n,m} = -t_x \exp(-i\phi_x) \quad if\ \vec{d}_m - \vec{d}_n = \hat{x}a$$

$$H_{n,m} = -t_x \exp(+i\phi_x) \quad if\ \vec{d}_m - \vec{d}_n = -\hat{x}a$$

$$H_{n,m} = -t_y \exp(-i\phi_y) \quad if\ \vec{d}_m - \vec{d}_n = \hat{y}a$$

$$H_{n,m} = -t_y \exp(+i\phi_y) \quad if\ \vec{d}_m - \vec{d}_n = -\hat{y}a$$

All other elements are zero. How would you choose $\varepsilon, t_x, \phi_x, t_y$ *and* ϕ_y so as to obtain a good match to the dispersion relation for small energy E ?

(a) $$E = \frac{(\hbar k_x - qA_x)^2}{2m} + \frac{(\hbar k_y - qA_y)^2}{2m}$$

(Simply replace $p_x \to \hbar k_x$, $p_y \to \hbar k_y$)

(b) $$E = \varepsilon - t_x e^{+i(k_x a - \phi_x)} - t_x e^{-i(k_x a - \phi_x)} - t_y e^{+i(k_y a - \phi_y)} - t_y e^{-i(k_y a - \phi_y)}$$

$$= \varepsilon - 2t_x \cos\underbrace{(k_x a - \phi_x)}_{\equiv\, \theta_x} - 2t_y \cos\underbrace{(k_y a - \phi_y)}_{\equiv\, \theta_y}$$

$$\simeq \varepsilon - 2t_x(1 - \theta_x^2/2) - 2t_y(1 - \theta_y^2/2)$$

$$= (\varepsilon - 2t_x - 2t_y) + t_x(k_x a - \phi_x)^2 + t_y(k_y a - \phi_y)^2$$

Compare, $E = \dfrac{\hbar^2}{2ma^2}\left(\left(k_x a - \dfrac{qA_x a}{\hbar}\right)^2 + \left(k_y a - \dfrac{qA_y a}{\hbar}\right)^2\right)$

$$\phi_x = \frac{qA_x a}{\hbar}, \quad \phi_y = \frac{qA_y a}{\hbar}$$

$$t_x = t_y = \hbar^2/2ma^2$$

$$\varepsilon - 2t_x - 2t_y = 0 \Rightarrow \varepsilon = 4t_x = 4t_y$$

2 2

Problem 3: Consider a channel with two spin levels, described by a (2x2) Hamiltonian

$$[H] = \varepsilon_0 I + \varepsilon \sigma_x$$

connected to two anti-parallel contacts 1 and 2, one communicating exclusively with +x spins and the other with -x spins: $\Sigma_1 = (-i\alpha/\cancel{4})[I+\sigma_x]$ and $\Sigma_2 = (-i\alpha/\cancel{4})[I-\sigma_x]$. A small bias is applied so that over a small energy range around E = 0 where we can assume f_1=1 and f_2=0.

(a) Find the correlation function G^n.

(b) Find the number of electrons N and the net spin $\vec{S}$ per unit energy in the channel.

$$G = \left(EI - \varepsilon_0 I - \varepsilon\sigma_x + i\alpha I\right)^{-1}$$

$$= \begin{pmatrix} E-\varepsilon_0+i\alpha & -\varepsilon \\ -\varepsilon & E-\varepsilon_0+i\alpha \end{pmatrix}^{-1}$$

$$= \frac{1}{(E-\varepsilon_0+i\alpha)^2-\varepsilon^2}\begin{bmatrix} E-\varepsilon_0+i\alpha & \varepsilon \\ \varepsilon & E-\varepsilon_0+i\alpha \end{bmatrix}$$

$$= g_0 I + g_x \sigma_x$$

where $g_0 \equiv \dfrac{E-\varepsilon_0+i\alpha}{(E-\varepsilon_0+i\alpha)^2-\varepsilon^2}$, $g_x = \dfrac{\varepsilon}{(E-\varepsilon_0+i\alpha)^2-\varepsilon^2}$

$$G^n = G\,\Gamma_1\,G^\dagger$$

$$= (g_0 I + g_x\sigma_x)\left(\alpha(I+\sigma_x)\right)(g_0^* I + g_x^*\sigma_x)$$

$$= \alpha(g_0 I + g_x\sigma_x)\left[g_0^* I + g_x^*\sigma_x + g_0^*\sigma_x + g_x^* I\right]$$

$$= \alpha \Big[\left(g_0 g_0^* + g_0 g_x^* + g_x g_x^* + g_x g_0^*\right) I + \left(g_x g_0^* + g_0 g_x^* + g_0 g_0^* + g_x g_x^*\right) \sigma_x \Big]$$

$$= \alpha \; (g_0 + g_x)(g_0^* + g_x^*) \; (I + \sigma_x)$$

$$g_0 + g_x = \frac{E - \varepsilon_0 + \varepsilon + i\alpha}{(E - \varepsilon_0 + i\alpha)^2 - \varepsilon^2} = \frac{1}{E - \varepsilon_0 - \varepsilon + i\alpha}$$

$$G^n = \frac{\alpha}{(E - \varepsilon_0 - \varepsilon)^2 + \alpha^2} \underbrace{(I + \sigma_x)}_{= \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}}$$

$$N = \mathrm{Trace}\left(\frac{G^n}{2\pi}\right) = \frac{\alpha/\pi}{(E - \varepsilon_0 - \varepsilon)^2 + \alpha^2}$$

$$S_x = \mathrm{Trace}\left(\frac{G^n \sigma_x}{2\pi}\right) = \frac{\alpha/\pi}{(E - \varepsilon_0 - \varepsilon)^2 + \alpha^2} = N$$

$$S_{y,z} = \mathrm{Trace}\left(\frac{G^n \sigma_{y,z}}{2\pi}\right) = 0$$

Problem 4: Consider a device (temperature T) with two sharp energy levels ε_1 - ε_2 = $\hbar\omega$, with the upper one in equilibrium with contact 1 and the lower one in equilibrium with contact 2. The current through this device can be written as

$$I \quad = \quad I_0\,[F(-\hbar\omega)\,f_1(\varepsilon_1)(1-f_2(\varepsilon_2) \quad - \quad F(+\hbar\omega)\,f_2(\varepsilon_2)(1-f_1(\varepsilon_1)]$$

where the absorption and emission rates characteristic of the surroundings (at temperature T_s) are given by $\qquad F(+\hbar\omega)=\dfrac{1}{\exp(\hbar\omega/kT_s)-1}$ $\quad$ and $\quad F(-\hbar\omega)=\exp(\hbar\omega/kT_s)\,F(+\hbar\omega)$

and the contact Fermi functions are given by $f_1(E)=f_0(E-\mu_1)$ and $f_2(E)=f_0(E-\mu_2)$

where $f_0(E)\equiv\dfrac{1}{\exp(E/kT)+1}$, $\mu_1=E_f+(qV/2)$, $\mu_2=E_f-(qV/2)$

(a) Show that the current can be written in the form:

$$I=I_0\,F(-\hbar\omega)\,(1-f_0(\varepsilon_1-\mu_1))\,f_0(\varepsilon_2-\mu_2)\,g(qV,\hbar\omega,kT,kT_s)$$

What is the function $g(qV,\hbar\omega,kT,kT_s)$?

(b) What is the open circuit voltage in terms of $\hbar\omega, kT, kT_s$?

$$I = I_0\,F(-\hbar\omega)\,(1-f_1(\varepsilon_1))\,f_2(\varepsilon_2)\left[\frac{f_1(\varepsilon_1)}{1-f_1(\varepsilon_1)}\,\frac{1-f_2(\varepsilon_2)}{f_2(\varepsilon_2)} - \frac{F(+\hbar\omega)}{F(-\hbar\omega)}\right]$$

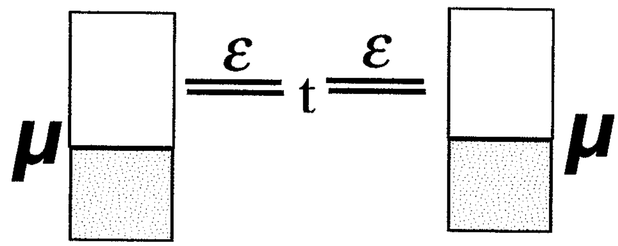
$$= I_0\,F(-\hbar\omega)\,(1-f_0(\varepsilon_1-\mu_1))\,f_0(\varepsilon_2-\mu_2)\,\Big[\underbrace{e^{-(\varepsilon_1-\mu_1)/kT}\,e^{(\varepsilon_2-\mu_2)/kT} - e^{-\hbar\omega/kT_s}}\Big]$$

$$= e^{(qV-\hbar\omega)/kT} - e^{-\hbar\omega/kT_s}$$

If $I=0$, $\qquad \dfrac{qV_{oc}-\hbar\omega}{kT} = \dfrac{-\hbar\omega}{kT_s} \Rightarrow V_{oc}=\dfrac{\hbar\omega}{q}\left(1-T/T_s\right)$

Problem 5: Consider two coupled quantum dots each with two spin-degenerate levels, such that the one-electron Hamiltonian, h

$$
\begin{array}{cc}
 & \begin{array}{cccc} a & b & \bar{a} & \bar{b} \end{array} \\
h = & \begin{bmatrix} \varepsilon & t & 0 & 0 \\ t & \varepsilon & 0 & 0 \\ 0 & 0 & \varepsilon & t \\ 0 & 0 & t & \varepsilon \end{bmatrix}
\end{array}
$$



Dot 'a' has an extremely high interaction energy so that the state $a\bar{a}$ is inaccessible in our energy range of interest and can be ignored. Only five (instead of the usual six) two-electron states need to be considered. Dot 'b' has zero interaction energy so that all five states $b\bar{b}, a\bar{b}, b\bar{a}, ab, \bar{a}\bar{b}$ have the same diagonal element 2ε.

What is the value of μ (at low temperatures) at which the number of electrons inside the coupled quantum dot system will change from N = 1 to N = 2 ?

$$
\begin{array}{ccc|cc}
b\bar{b} & a\bar{b} & b\bar{a} & ab & \bar{a}\bar{b} \\
2\varepsilon & t & t & 0 & 0 \\
t & 2\varepsilon & 0 & 0 & 0 \\
t & 0 & 2\varepsilon & 0 & 0 \\
\hline
0 & 0 & 0 & 2\varepsilon & 0 \\
0 & 0 & 0 & 0 & 2\varepsilon
\end{array}
$$

$$
\begin{array}{ccc}
b\bar{b} & \dfrac{a\bar{b}+b\bar{a}}{\sqrt{2}} & \dfrac{a\bar{b}-b\bar{a}}{\sqrt{2}}
\end{array}
$$

$$
\begin{pmatrix} 2\varepsilon & \sqrt{2}\,t & 0 \\ \sqrt{2}\,t & 2\varepsilon & 0 \\ 0 & 0 & 2\varepsilon \end{pmatrix}
$$

Eigenenergies: $2\varepsilon, 2\varepsilon, 2\varepsilon, 2\varepsilon \pm \sqrt{2}\, t$

Lowest : $2\varepsilon - \sqrt{2}\, |t|$

1-electron state, lowest energy : $\varepsilon - |t|$

$$\varepsilon - |t| - \mu = 2\varepsilon - \sqrt{2}\,|t| - 2\mu$$

$$\boxed{\mu = \varepsilon - |t|\,(\sqrt{2} - 1)}$$