Practice Exam 2 Solution

Problem 1

$$P_{0000} = \frac{1}{Z} \exp(0)$$

$$P_{1000} = \frac{1}{Z} \exp\left(-\left((1)\varepsilon - \mu(1)\right)/kT\right) = \frac{1}{Z} \exp\left(-(\varepsilon-\mu)/kT\right)$$

$$= P_{0100} = P_{0010} = P_{0001}$$

$$P_{0000} + 4P_{1000} = 1$$

$$\left(1 + 4\exp\left(-(\varepsilon-\mu)/kT\right)\right) = Z$$

$$\therefore N_{average} = 0 \cdot P_{0000} + 4P_{1000}$$

$$= \frac{4\exp\left(-(\varepsilon-\mu)/kT\right)}{1 + 4\exp\left(-(\varepsilon-\mu)/kT\right)}$$

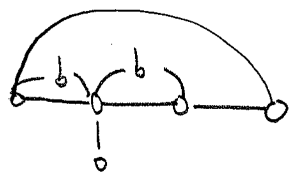
$$= \frac{1}{1 + \frac{1}{4}\exp\left(\frac{\varepsilon-\mu}{kT}\right)}$$

Problem 2

$$\begin{bmatrix} 0 & b & 0 & b \\ b & 0 & b & 0 \\ 0 & b & 0 & b \\ b & 0 & b & 0 \end{bmatrix}$$

From the principles of bandstructure

$$\phi = \phi_0 e^{ikna}$$



$$E = 0 + b e^{ika} + b e^{-ika} = 2b \cos ka$$

Because of periodic boundary condition

$$e^{i4ka} = 1 \Rightarrow ka = \frac{2\pi n}{4} = \frac{\pi n}{2} \quad (n: \text{Integer})$$

ka :	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$
E (eigenvalues) :	$2b$	0	$-2b$	0

eigenvectors :

$$\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \quad \begin{pmatrix} 1 \\ i \\ -1 \\ -i \end{pmatrix} \quad \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} \quad \begin{pmatrix} 1 \\ -i \\ -1 \\ i \end{pmatrix} \quad /2$$

Problem 3

$$\varepsilon(\vec{k}) = \alpha\sqrt{k_x^2+k_y^2} = \alpha k \qquad (\alpha > 0)$$

$$(d\varepsilon = \alpha\, dk)$$

$$D(E) = \sum_k \delta(E - \varepsilon(\vec{k}))$$

$$= \iint \frac{dk_x\, dk_y}{\frac{2\pi}{L}\frac{2\pi}{W}} \delta(E - \varepsilon(\vec{k}))$$

$$= \frac{LW}{4\pi^2}\int_{-\pi}^{\pi} d\theta \int dk\, k\ \delta(\varepsilon(\vec{k}) - E)$$

$$= \frac{LW}{4\pi^2}\ 2\pi \int \frac{d\varepsilon}{\alpha}\ \frac{\varepsilon}{\alpha}\ \delta(\varepsilon(\vec{k}) - E)$$

$$= \frac{LW}{2\pi}\ \frac{1}{\alpha^2}\ E = \frac{LW}{2\pi\alpha^2} E\,\theta(E)$$

$$M(E) = \sum_k \delta(E - \varepsilon(\vec{k}))\ \frac{\pi\hbar|v_x(\vec{k})|}{L} \qquad \left(v_x = \frac{\partial\varepsilon}{\hbar\,\partial k_x} = \frac{\alpha}{\hbar}\frac{k_x}{k} = \frac{\alpha}{\hbar}\cos\theta\right)$$

$$= \frac{\cancel{L}W}{4\pi^2}\int_{-\pi}^{\pi} d\theta \int dk\, k\ \delta(E - \varepsilon(k))\ \frac{\pi\cancel{\hbar}}{\cancel{L}}\ \frac{\alpha}{\cancel{\hbar}}|\cos\theta|$$

$$= \alpha\frac{W}{4\pi}\ 2\cdot\int_{-\pi/2}^{\pi/2} d\theta\ \cos\theta \int_0^\infty \frac{d\varepsilon}{\alpha}\ \frac{\varepsilon}{\alpha}\ \delta(E - \varepsilon)$$

$$= \frac{W}{\pi\alpha} E\,\theta(E)$$