Practice Final Solution.

Problem 1

$$D(E) = A E^{\alpha}$$

$$N = \int_0^{\mu} dE\, D(E) = A \frac{E^{\alpha+1}}{\alpha+1}\Big|_0^{\mu} = \frac{A}{\alpha+1} \mu^{\alpha+1}$$

$$M(E) = B E^{\beta} \Rightarrow M(\mu) = B\mu^{\beta} = B\left(\frac{(\alpha+1)N}{A}\right)^{\frac{\beta}{\alpha+1}}$$

At T=0K,

$$G(\mu) = \frac{q^2}{h} M(\mu), \text{ (per spin)}$$

$$= \frac{q^2}{h} B \left(\frac{\alpha+1}{A}\right)^{\frac{\beta}{\alpha+1}} N^{\frac{\beta}{\alpha+1}} = G(N)$$

Problem 2

(a)

$$\begin{cases} P_{01} + P_{10} + P_{00} = 1 \\ P_{10} = P_{01} \\ \Gamma_2 P_{01} = \Gamma_1 P_{00} \quad \text{(Steady state)} \end{cases} \Rightarrow P_{00}, P_{01}, P_{10}$$

we can find

$$I = \frac{q}{\hbar} 2\Gamma_2 P_{01} = \frac{q}{\hbar} 2\Gamma_2 \left(\frac{\Gamma_1}{2\Gamma_1 + \Gamma_2}\right) = \frac{q}{\hbar}\frac{2\Gamma_1\Gamma_2}{2\Gamma_1+\Gamma_2}$$

(b) $(\Gamma_1 = 10\Gamma_2)$

$$I_1 = \frac{q}{\hbar}\frac{2\cdot 10\,\Gamma_2^{\cancel{2}}}{21\cancel{\Gamma_2}} = \frac{q}{\hbar}\frac{20}{21}\Gamma_2$$

In a reverse case, roles of Γ_1 and Γ_2 are interchanged.

$$I_2 = \frac{q}{\hbar}\frac{2\Gamma_1\Gamma_2}{\Gamma_1+2\Gamma_2} = \frac{q}{\hbar}\frac{20\Gamma_2^2}{12\Gamma_2} = \frac{q}{\hbar}\frac{20}{12}\Gamma_2 \quad \text{(more current)}$$

Problem 3

Assuming $\phi_n = \phi_0 e^{ikna}$ to solve $E\phi = H\phi$.

Every row gives the following equation

$$E\phi_n = \varepsilon\phi_n + t e^{-i\varphi}\phi_{n-1} + t e^{i\varphi}\phi_{n+1}$$

$$\Rightarrow E = \varepsilon + t e^{-i\varphi} e^{-ika} + t e^{i\varphi} e^{+ika}$$

$$= \varepsilon + 2t\cos(\varphi + ka)$$

Problem 4

$$\varepsilon = Bk^2 \quad (3D) \qquad d\varepsilon = 2Bk\,dk$$
$$= B(k_x^2 + k_y^2 + k_z^2)$$

$$D(E) = \sum_{\vec{k}} \delta(E - \varepsilon(k))$$

solid angle integration.

$$= \frac{S\cdot L}{(2\pi)^3}\int_0^\infty 4\pi\, k^2 dk\; \delta(E-\varepsilon)$$

$$= \frac{S\cdot L}{(2\pi)^3} 4\pi \int_0^\infty \frac{d\varepsilon}{2B\cancel{k}} k^{\cancel{2}} \delta(E-\varepsilon)$$

$$= \frac{S\cdot L}{2\pi^2}\int_0^\infty \frac{1}{2B}\sqrt{\frac{\varepsilon}{B}}\, d\varepsilon\, \delta(E-\varepsilon)$$

$$= \frac{S\cdot L}{4\pi^2 B^{3/2}}\sqrt{E} \qquad \text{(per spin)}$$

$$\text{DOS} / \text{Volume } (S\cdot L) = \frac{1}{4\pi^2 B^{3/2}}\sqrt{E}$$

$$M(E) = \sum_{\vec{k}} \delta(E - \varepsilon(k)) \frac{\pi \hbar |v_z(\vec{k})|}{L}$$

$$v_z = \frac{1}{\hbar}\frac{\partial \varepsilon}{\partial k_z} = \frac{1}{\hbar} 2Bk_z = \frac{2B}{\hbar} k \cos\theta$$

$$= \frac{S \cdot \cancel{L}}{(2\pi)^3} \int_0^\infty k^2 dk \int_0^\pi \sin\theta \, d\theta \int_0^{2\pi} d\varphi \, \delta(E-\varepsilon) \frac{\pi 2B|k\cos\theta|}{\cancel{L}}$$

$$= \frac{\pi 2BS}{(2\pi)^3} \int_0^\infty k^3 dk \int_0^\pi \sin\theta |\cos\theta| d\theta \cdot 2\pi \cdot \delta(E-\varepsilon)$$

$$= \frac{\cancel{2\pi B}S}{(2\pi)^{\cancel{3}\,2}} \int_0^\infty \frac{k^{\cancel{3}\,2}}{\cancel{2Bk}} d\varepsilon \underbrace{\int_0^{\pi/2} \cancel{2} \sin\theta\cos\theta \, d\theta}_{\parallel} \delta(E-\varepsilon)$$

$$= \frac{S}{2\pi} \int_0^\infty \frac{\varepsilon}{B} \delta(E-\varepsilon) d\varepsilon \cdot \frac{1}{2}$$

$$= \frac{S}{4\pi B} E$$

Problem 5

Spins along $+\vec{n} = \begin{Bmatrix} c \\ s \end{Bmatrix} = \begin{Bmatrix} \cos\frac{\theta}{2}\, e^{-i\varphi/2} \\ \sin\frac{\theta}{2}\, e^{+i\varphi/2} \end{Bmatrix}$, $-\vec{n} = \begin{Bmatrix} -s^* \\ c^* \end{Bmatrix}$.

Appling basis transformation

$$\begin{bmatrix} c & -s^* \\ s & c^* \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} c^* & s^* \\ -s & c \end{bmatrix}$$

$$= \begin{bmatrix} c & s^* \\ s & -c^* \end{bmatrix} \begin{bmatrix} c^* & s^* \\ -s & c \end{bmatrix}$$

$$= \begin{bmatrix} cc^* - ss^* & 2cs^* \\ 2c^*s & ss^* - c^*c \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & \sin\theta\, e^{-i\varphi} \\ \sin\theta\, e^{+i\varphi} & -\cos\theta \end{bmatrix}$$

$$= \cos\theta \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}}_{\sigma_z} + \sin\theta\cos\varphi \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{=\sigma_x}$$

$$+ \sin\theta\sin\varphi \underbrace{\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}}_{=\sigma_y}$$

Problem 6

Treat "0" point as device, rest as self-energy

(a) $H = [2t_0]$

$$\Sigma_1 = -t_0 e^{ika} \quad , \quad \Sigma_2 = -t_0 e^{ika}$$

$$G = [E - 2t_0 + 2t_0 e^{ika}]^{-1}$$

$$= \frac{1}{2t_0} \frac{1}{e^{ika} - \cos ka} = \frac{-i}{2t_0 \sin ka}$$

$$D = \frac{A}{2\pi} = \frac{i}{2\pi}\left(\frac{-2i}{2t_0 \sin ka}\right) = \frac{1}{2\pi t_0 \sin ka}$$

$$\cos ka = \frac{2t_0 - E}{2t_0} \quad , \quad \sin ka = \sqrt{1 - \left(\frac{2t_0 - E}{2t_0}\right)^2} = \sqrt{(4t_0 - E)E}$$

$$\therefore D(0,E) = \frac{A}{2\pi} = \frac{1}{\pi\sqrt{(4t_0 - E)E}}$$

(b) here $\Sigma_2 = 0$

$$G = \frac{1}{t_0} \frac{1}{e^{ika} - 2\cos ka} = -\frac{1}{t_0} e^{ika}$$

$$D(0,E) = \frac{A}{2\pi} = \frac{i}{2\pi}\left(-\frac{1}{t_0}\right) 2i \sin ka$$

$$= \frac{\sin ka}{\pi t_0} = \frac{\sqrt{(4t_0 - E)E}}{2\pi t_0^2}$$