Fall 2009 ECE 495N FINAL EXAM

CLOSED BOOK

Friday, Dec.19, 7P-9P, LYNN G165

NAME : SOLUTION

PUID # : ______________________

Please show all work and write your answers clearly.

This exam should have eight pages + 1 page of notes.

Problem 1	[p. 2]	5 points
Problem 2	[p. 3]	5 points
Problem 3	[p. 4]	5 points
Problem 4	[p. 5]	5 points
Problem 5	[p. 6]	5 points
Problem 6	[p. 7,8]	5 points
Total		30 points

Problem 1: Suppose a 2-D conductor of length L and width W has a density of states (per unit energy), $D(E) = (D_0 LW)\, E$ and a density of modes (dimensionless), $M(E) = (M_0 W) E$, where D_0 and M_0 are constants. Assuming ballistic transport, obtain an expression for the conductance per unit width at T=0K, as a function of the electron density per unit area.

$$N = \int_0^{\mu} (D_0 L W)\, E\, dE = (D_0 L W)\, \mu^2/2$$

$$n_s = D_0\, \mu^2/2$$

$$G = \frac{q^2}{h} M(\mu) = \frac{q^2}{h} M_0 W \cdot \mu.$$

$$\frac{G}{W} = \frac{q^2 M_0}{h} \mu$$

$$= \frac{q^2 M_0}{h} \sqrt{\frac{2 n_s}{D_0}}$$

Problem 2: A channel has four energy levels all with the same energy ε, but the interaction energy is so high that no more than one of these levels can be occupied at the same time. Starting from the law of equilibrium, what is the average number of electrons in the channel if it is in equilibrium with chemical potential μ and temperature T? Your answer should be in terms of ε, μ and T.

$$P_0 + 4P_1 = 1 \qquad x = \frac{\varepsilon - \mu}{kT}$$

$$P_0 \rightarrow \frac{1}{Z}, \qquad P_1 \rightarrow \frac{1}{Z} e^{-x}$$

$$Z = \frac{1}{1 + 4e^{-x}}$$

$$N = 4P_1 = \frac{4e^{-x}}{1 + 4e^{-x}}$$

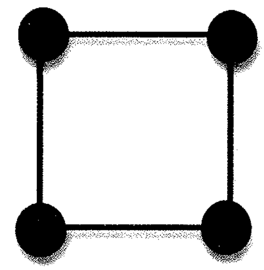
$$= \frac{1}{1 + \frac{1}{4} e^{(\varepsilon - \mu)/kT}}$$

Problem 3: A molecule (NOT a solid) consists of four atoms arranged at the corners of a square. Assume (1) one orbital per carbon atom as basis function ; (2) the overlap matrix [S] is a (4x4) identity matrix; and (3) the Hamiltonian matrix is given by

$H_{n,n} = \varepsilon$ (site energy)

$H_{n,m} = t$ if n, m are neighboring atoms

$H_{n,m} = 0$ if n, m are NOT nearest neighbors



What are the four energy eigenvalues in terms of 'ε' and 't' and the corresponding eigenvectors. ***Please do not diagonalize directly: use the principle of bandstructure.***

$$E = \varepsilon + 2t \cos(\phi)$$

$$\phi = -\frac{\pi}{2},\ 0,\ +\frac{\pi}{2},\ +\pi$$

$$= \varepsilon,\ \varepsilon + 2t,\ \varepsilon,\ \varepsilon - 2t$$

$$\begin{pmatrix} 1 \\ e^{i\phi} \\ e^{i2\phi} \\ e^{i3\phi} \end{pmatrix}$$

Problem 4: Suppose a ***large*** two-dimensional conductor has an $\varepsilon(\vec{k})$ relationship given by:

$$\varepsilon(\vec{k}) = +\alpha\sqrt{k_x^{\,2} + k_y^{\,2}}$$

where α is a constant. Derive an expression for the density of modes, M(E). Your answer should be in terms of the energy E, α, width W and length L. ($v_x(\vec{k}) = \frac{1}{\hbar}\frac{\partial \varepsilon}{\partial k_x}$)

$$M(E) = \sum_{\vec{k}} \delta(\varepsilon - E)\,\frac{\pi}{L}\,\alpha\,|\cos\theta|$$

$$= \frac{LW}{4\pi^2}\int k\,dk \int_{-\pi}^{+\pi} d\theta\;\delta(\varepsilon - E)\,\frac{\pi\alpha}{L}\cos\theta$$

$$= \frac{\cancel{L}W}{2\pi^{\cancel{2}}}\int_{-\pi/2}^{+\pi/2}\cos\theta \int \frac{d\varepsilon}{\alpha}\,\frac{\varepsilon}{\cancel{\alpha}}\,\delta(\varepsilon - E)\,\frac{\cancel{\pi}\cancel{\alpha}}{\cancel{L}}$$

$$= \frac{W}{\cancel{2}\pi\alpha}\cdot\cancel{2}E$$

Problem 5: Early in this course we showed that the maximum current through a device with a single level with escape rates γ_1, γ_2 is given by $(q/\hbar)\,\gamma_1\gamma_2/(\gamma_1+\gamma_2)$ using elementary arguments. Prove this result using the general matrix expressions discussed later in the course starting from a (1x1) Hamiltonian and contact self-energies

$$[H]=[\varepsilon] \qquad [\Sigma_1]=-i\,[\gamma_1/2] \ , \ [\Sigma_2]=-i\,[\gamma_2/2]$$

(***Useful relation:*** $\int\limits_{-\infty}^{\infty} \frac{\gamma\, dE}{E^2+(\gamma/2)^2} = 2\pi$)

$$\text{Trace}\,[\Gamma_1 G \Gamma_2 G^\dagger]$$

$$= \gamma_1 \frac{1}{E-\varepsilon+i\frac{\gamma_1+\gamma_2}{2}}\ \gamma_2 \frac{1}{E-\varepsilon-i\frac{\gamma_1+\gamma_2}{2}}$$

$$= \frac{\gamma_1\gamma_2}{\underbrace{\gamma_1+\gamma_2}_{\equiv\gamma}} * \frac{\gamma}{(E-\varepsilon)^2+(\gamma/2)^2}$$

$$I = \frac{q}{h}\int dE\ \frac{\gamma}{(E-\varepsilon)^2+(\gamma/2)^2}\cdot\frac{\gamma_1\gamma_2}{\gamma}\left[\overset{1}{\cancel{f_1}}-\overset{0}{\cancel{f_2}}\right]$$

$$= \frac{q}{\cancel{h}\,\hbar} * \cancel{2\pi} * \frac{\gamma_1\gamma_2}{\gamma_1+\gamma_2}$$

Problem 6: Consider a device with two spin-degenerate levels described by $[H]=\begin{bmatrix}\varepsilon & 0\\ 0 & \varepsilon\end{bmatrix}$ and with one contact magnetized along +z with $[\Gamma_1]=\begin{bmatrix}\alpha & 0\\ 0 & 0\end{bmatrix}$. Contact 2 is identical to contact 1, except that it is magnetized along +x instead of +z. What is the corresponding $[\Gamma_2]$?

Hint: If we use up- and down-spins along +x as our basis, $[\Gamma_2]$ ***would look just like*** $[\Gamma_1]$***. Now transform this into the regular basis using up- and down-spins along*** $\hat{z}$***.***

Spins along $+\hat{n}:\{c \quad s\}^T,\ -\hat{n}:\{-s^* \quad c^*\}^T$ ***(T:transpose)***

where $c \equiv \cos(\theta/2)\, e^{-i\varphi/2}$ ***and*** $s \equiv \sin(\theta/2)\, e^{+i\varphi/2}$

$$\Gamma_2 = \frac{1}{2}\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}\begin{bmatrix}\alpha & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}1 & +1\\ -1 & 1\end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}\begin{bmatrix}\alpha & +\alpha\\ 0 & 0\end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix}\alpha & \alpha\\ \alpha & \alpha\end{bmatrix} = \frac{\alpha}{2}\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}$$