Lecture 9: Coupled Current Equations

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charge and heat currents

1D, 2D, or 3D conductor

\[ I = G \Delta V - [SG] \Delta T \]

\[ I_Q = T[SG] \Delta V - K_0 \Delta T \]

driving “forces:”

1) differences in voltage (electrochemical potential)

2) differences in temperature
signs of the coefficient

\[ I = G \Delta V - [SG] \Delta T \]

\[ I_Q = T[SG] \Delta V - K_0 \Delta T \]

\( G > 0 \)

\([SG] > 0 \)

\( K_0 > 0 \)

(electronic) thermal conductance for \( \Delta V = 0 \)
conductance vs. conductivity

\[ T_2 > T_1 \]

\[ IQ = T[S\sigma] \Delta V - K_0 \Delta T \]

3D conductors

\[ G = \sigma \frac{A}{L} \]

\[ K_0 = \kappa_0 \frac{A}{L} \]

etc.
The proper driving forces are changes in the electrochemical potential and in the inverse temperature.
coupled flows

\[ I = L_{11} \Delta (E_F) + L_{12} \Delta \left( \frac{1}{T} \right) \]

\[ I_Q = L_{21} \Delta E_F + L_{22} \Delta \left( \frac{1}{T} \right) \]

\[ I_1 = L_{11} \left( \begin{array}{c} 1 \\ B \end{array} \right) F_1 + L_{12} \left( \begin{array}{c} 1 \\ B \end{array} \right) F_2 \]

\[ I_2 = L_{21} \left( \begin{array}{c} 1 \\ B \end{array} \right) F_1 + L_{22} \left( \begin{array}{c} 1 \\ B \end{array} \right) F_2 \]

\[ I_1, I_2 \quad \text{“generalized fluxes”} \]

\[ F_1, F_2 \quad \text{“generalized forces”} \]

\[ L_{12} = L_{21} \quad \text{Onsager relation} \]
temperature differences produce heat currents

pressure differences produce matter currents

heat flow per pressure difference = matter flow per temperature difference


http.en.wikipedia.org/wiki/Onsager_reciprocal_relations
for more about this topic


Irreversible thermodynamics: Chapter 2

Onsager relations: Chapter 3
1) Onsager relations

2) Measurement considerations

3) Thermoelectric devices
measurements

\[ I = G\Delta V - [SG]\Delta T \]
\[ I_Q = T[SG]\Delta V - K_0\Delta T \]

\[ \Delta V = RI - S\Delta T \]
\[ I_Q = -\pi I - K_\varepsilon\Delta T \]
two forms of the equations

\[ I = G \Delta V - [SG] \Delta T \]
\[ I_Q = T[SG] \Delta V - K_0 \Delta T \]

\[ \Delta V = RI - S \Delta T \]
\[ I_Q = -\pi I - K_e \Delta T \]

\[ \pi = TS < 0 \]
\[ K_e = K_0 - T \frac{[SG]^2}{G} > 0 \]

electronic thermal conductance for \( I = 0 \)

\[ \Delta V = \frac{1}{G} I + \frac{[SG]}{G} \Delta T \]
\[ R = 1/G \quad S = -[SG]/G \]

\[ \Delta V = RI - S \Delta T \]
\[ I_Q = T[SG](RI - S \Delta T) - K_0 \Delta T \]

\[ I_Q = T \frac{[SG]}{G} I - (K_0 + T[SG]S) \Delta T \]
\[ I_Q = -TSI - \left( K_0 - T \frac{[SG]^2}{G} \right) \Delta T \]
resistivity / conductivity measurements

\[ I = G \Delta V - [SG] \Delta T \]
\[ I_Q = T[SG] \Delta V - K_0 \Delta T \]

\[ \Delta V = RI - S\Delta T \]
\[ I_Q = -\pi I - K_c \Delta T \]

People generally measure **resistivity** (or **conductivity**) because for bulk materials, these parameters depend on material properties and not on the length of the resistor or its width or cross-sectional area.
resistance and resistivity

How does the resistivity depend on the parameters of the semiconductor?

\[ I = \frac{V}{R} \]

\[ R \propto \frac{L}{A} \, \Omega \]

\[ R = \rho \frac{L}{A} \, \Omega \]

resistivity:

\[ \rho \, \frac{1}{2}-m \]
conductance and conductivity

\[ I = GV = V/R \]

\[ G = \frac{1}{R} \propto \frac{A}{L} S \left( \Omega \right)^{-1} \]

\[ G = \sigma \frac{A}{L} \]

conductivity:

\[ \sigma = \frac{1}{\rho} \text{ S/m} \]
traditional conductance and conductivity

cross-sectional area, $A$

'ideal' contacts

n-type semiconductor

$I = GV = \frac{Q}{\tau}$ Amperes

$\langle \nu \rangle = \mu_n E = \mu_n \frac{V}{L}$ cm/sec

$\tau = \frac{L}{\langle \nu \rangle} = \frac{L^2}{\mu_n V}$

$I = \left\{ (nq\mu_n) \frac{A}{L} \right\} V = GV$

$G = \sigma \frac{A}{L}$

$\sigma = nq\mu_n$
Landauer conductance and conductivity

\[ I = GV \]

\[ G \propto \frac{A}{L} S \left( \Omega \right)^{-1} \]

\[ G = \sigma \frac{A}{L} \quad \sigma = \frac{1}{\rho} \quad \text{S/m} \]

\[ \sigma = \frac{G}{A/L} = \frac{2q^2}{h} \left( \int \frac{\lambda(E) M(E)}{A} \left( -\frac{\partial f_0}{\partial E} \right) dE \right) \]

\[ \left( T(E) \approx \lambda(E)/L \right) \]
sheet conductance

\[ A = Wt \]

\[ I = GV \]

\[ G = \sigma \frac{A}{L} \quad \sigma = nq\mu_n \]

\[ G = \sigma \left( \frac{Wt}{L} \right) = nq\mu_n t \left( \frac{W}{L} \right) \]

\[ G = \sigma_s \left( \frac{W}{L} \right) \]

\[ \sigma_s = n_s q \mu_n \]

‘sheet conductance’
2-probe measurements

\[
R_{DEV} = \rho_s \frac{L}{W}
\]

\[
V_{21} = I \left( 2R_C + R_{DEV} \right)
\]

\[
R_{DEV} \neq \frac{V_{21}}{I}
\]
4-probe measurements

Top view

\[ R_{DEV} = \rho_s \frac{L}{W} \]

\[ V_{21} = I \left( R_{DEV} \right) \]

(high impedance voltmeter)

“Hall bar geometry”
van der Pauw technique

Force $I$ through two contacts, measure $V$ between the other two contacts.

$$R_{1,4/2,3} \equiv \frac{V_{23}}{I_{14}} \quad R_{2,1/3,4} \equiv \frac{V_{34}}{I_{21}}$$

$$e^{-\frac{\pi}{\rho_S} R_{1,4/2,3}} + e^{-\frac{\pi}{\rho_S} R_{2,1/3,4}} = 1$$

$$\Rightarrow \rho_S$$
isothermal vs. adiabatic

\[ R_{\text{DEV}} = \rho S \frac{L}{W} \]

\[ V_{21} = R_{\text{DEV}} I - S \Delta T \]

\[ I_Q = -\pi I - K_e \Delta T \]

\( \hat{i}) \quad \Delta T = 0 \quad V_{21} = R_{\text{DEV}} I \rightarrow R_{\text{DEV}} = \frac{V_{21}}{I} \)

\( \hat{ii}) \quad I_Q = 0 \quad \Delta T = \left(-\frac{\pi}{K_e}\right)I \quad V_{21} = R_{\text{DEV}} I + \left(S^2 T/K_e\right)I \quad V_{21}/I = R_{\text{DEV}} + \left(S^2 T/K_e\right) \)
for more about low-field measurements


Lundstrom, Chapter 4, Sec. 7
outline

1) Onsager relations
2) Measurement considerations
3) Thermoelectric devices
thermoelectric devices: cooling

N-type P-type

cool

figure of merit:

\[ ZT = \frac{S^2 GT}{K} \]
thermoelectric devices: power generation

figure of merit:

$$ZT = \frac{S^2 GT}{K}$$
the TE industry: 2009

1) NASA: Radioisotope Thermoelectric Generators for deep-space missions.

2) DOD: cooling for lasers, night vision, sensors, guidance systems, etc.

1) World market for commercial TE power generation: $30-50M/yr

2) World market for commercial TE cooling: $200-250M/yr

Automobile market (seat cooler/heater) is growing. Vehicle waste heat recovery to improve fuel efficiency is a potential ‘killer-app.’
If we heat contact 1, and keep contact 2 cool, what is the maximum efficiency with which we can heat into useful work?

**Answer:**

**Maximum efficiency:**

\[ \eta_{\text{max}} = 1 - \frac{T_2}{T_1} = 1 - \frac{T_{\text{COLD}}}{T_{\text{HOT}}} \]

Carnot’s theorem
what’s necessary for current to flow?

\[ I(E) = \frac{2q}{h} T(E) M(E) (f_1 - f_2) \]

Current flows when there is a difference in Fermi levels.

\[ f_1(E) = \frac{1}{1 + e^{(E - E_{F1})/k_B T_1}} \]

\[ f_2(E) = \frac{1}{1 + e^{(E - E_{F2})/k_B T_2}} \]

\[ I(E) > 0 \Rightarrow f_1 > f_2 \]

\[ \Rightarrow \frac{(E - E_{F1})}{k_B T_1} < \frac{(E - E_{F2})}{k_B T_2} \]
maximum efficiency of a TE heat engine

Thermal energy absorbed from contact 1:

$$Q_1 = (E - E_{F1})$$

Thermal energy dissipated into contact 2:

$$Q_2 = (E - E_{F2})$$

Net thermal energy absorbed from contact 1 that can be converted into useful work:

$$Q_1 - Q_2$$

Efficiency:

$$\eta = \frac{Q_1 - Q_2}{Q_1} = 1 - \frac{Q_2}{Q_1} = 1 - \frac{Q_2}{Q_1} \left( \frac{T_2}{T_1} \right)$$

Carnot’s theorem
If we force a current through this thermoelectric device, what is the maximum temperature difference, \( \Delta T = T_2 - T_1 \), that we can produce?
maximum temperature difference

\[ \Delta V = RI - S\Delta T \]

\[ I_Q = -\pi I - K_e\Delta T \]

\[ I_{Q1} = -TSI - K_e\Delta T - \frac{1}{2} I^2 R \]

For a fixed current, the temperature will rise until \( I_Q = 0 \)

\[ \Delta T = -\frac{(TSI + I^2 R/2)}{K_e} \]

\[ \frac{\partial \Delta T}{\partial I} = -\frac{(TS + IR)}{K_e} = 0 \]

\[ I_{opt} = -\frac{TS}{R} \]

\[ \Delta T|_{\text{max}} = \frac{1}{2} \frac{T^2 S^2 G}{K} \]
maximum temperature difference

\[
\Delta T_{\text{max}} = \frac{1}{2} \frac{T^2 S^2 G}{K}
\]

\[
\frac{\Delta T}{T}_{\text{max}} = \frac{1}{2} \frac{S^2 GT}{K} = \frac{1}{2} ZT
\]

\[
ZT = \frac{S^2 GT}{K_e + K_L}
\]

dimensionless figure of merit

\(K_e\): electronic heat conductance

\(K_L\): lattice heat conductance
figure of merit

$$ZT = \frac{S^2 GT}{K_e + K_L}$$

**Power factor:**
- large Seebeck coefficient
- high conductance
- depends on material parameters

**thermal conductance:**
- $K_L > K_e$
- need short mfp for phonons
  (alloys with large mass difference – e.g. $\text{Bi}_2\text{Te}_3$, SiGe)
- nanostructuring for reducing mfp
- “phonon glass electron crystal”

The optimum power factor occurs when $E_F$ is near $E_C$. 

$$
\eta_F = \frac{S^2 G T}{k_B T}
$$

$$
PF = S(\eta_F)
$$

$$
PF = S^2 G T
$$

$$
\eta_F = \frac{(E_F - E_C)}{k_B T}
$$
history and prospects

\[ ZT = \frac{S^2 G T_L}{K_e + K_L} \]

- Bi$_2$Te$_3$ and alloys with Sb, Se
- CVD SL
- natural nano-dots
- Si NW
- nano-powder
- quantum dot SL

questions

1) Onsager relations
2) Measurement considerations
3) Thermoelectric devices