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Lecture 18: Strong magnetic fields

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Hall and integer quantum Hall effect

outline

1) Magnetoconductivity tensor
2) Resistivity tensor
3) Strong B-fields: Landau levels
4) Shubnikov-DeHaas Oscillations and QHE
5) Summary
**B-field dependent DD equation**

\[
\vec{F}_e = -q\vec{E} - q\vec{v} \times \vec{B} = \frac{d\vec{p}}{dt}
\]

\[
\vec{p} = \left( -q\vec{E} - q\vec{v} \times \vec{B} \right) \tau = m^* \vec{v}
\]

\[
\vec{v} \approx -\frac{q\tau}{m^*} \vec{E} - \frac{q\tau}{m^*} \vec{v} \times \vec{B}
\]

(Can be solved exactly for the velocity. See prob. 4.18, Lundstrom.)

\[
\vec{v} \approx -\frac{q\tau}{m^*} \vec{E}
\]

\[
\vec{v} \approx -\frac{q\tau}{m^*} \vec{E} + \frac{q^2\tau^2}{(m^*)^2} \vec{E} \times \vec{B}
\]

(Low B-fields)

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B-field dependent DD equation

\[ \vec{v} \approx -\frac{q \tau}{m^*} \vec{E} + \frac{q^2 \tau^2}{(m^*)^2} \vec{E} \times \vec{B} \]

\[ \langle \vec{v} \rangle \approx -\frac{q \langle \tau \rangle}{m^*} \vec{E} + \frac{q^2 \langle \tau^2 \rangle}{(m^*)^2} \vec{E} \times \vec{B} \]

\[ \vec{J}_n = -nq \langle \vec{v} \rangle = nq \left[ \frac{q \langle \tau \rangle}{m^*} \right] \vec{E} - nq \frac{q^2 \langle \tau^2 \rangle}{(m^*)^2} \vec{E} \times \vec{B} \]

\[ \vec{J}_n = nq \mu_n \vec{E} - nq \frac{q \langle \tau \rangle}{m^*} \frac{q \langle \tau \rangle}{m^*} \left( \frac{\langle \tau^2 \rangle}{\langle \tau \rangle^2} \right) \vec{E} \times \vec{B} \]

\[ \vec{J}_n = nq \mu_n \vec{E} - (\sigma_n \mu_n r_H) \vec{E} \times \vec{B} \]

\[ J_i = nq \mu_n E_i - \sigma_n \mu_H \varepsilon_{ijk} E_j B_k \]

\[ \mu_H = r_H \mu_n \]

\[ r_H = \frac{\langle \tau^2 \rangle}{\langle \tau \rangle^2} \]
\[ \vec{J}_n = nq\mu_n \vec{E} - (\sigma_n\mu_n r_H) \vec{E} \times \vec{B} \]

\[ \mu_H = r_H \mu_n \quad \text{“Hall mobility”} \]

\[ r_H = \frac{\left\langle \tau^2 \right\rangle}{\left\langle \tau \right\rangle^2} \quad \text{“Hall factor”} \]

(The BTE tells us how to properly compute the averages.)
The Hall effect was discovered by Edwin Hall in 1879 and is widely used to characterize electronic materials. It also finds use in magnetic field sensors.

\[ \vec{B} = B\hat{z} \]

- current in \( x \)-direction: \( I_x \)
- B-field in \( z \)-direction: \( \vec{B} = B\hat{z} \)
- Hall voltage measured in the \( y \)-direction: \( V_H > 0 \) (n-type)
example: Hall effect

\[ \vec{J}_n = nq \mu_n \vec{E} - \left( \sigma_n \mu_n r_H \right) \vec{E} \times \vec{B} \]

current in x-direction
measure voltage (electric field) in y-direction
B-field in z-direction

\[ J_x = nq \mu_n \vec{E}_x - \left( \sigma_n \mu_n r_H \right) \vec{E}_y B_z \approx nq \mu_n \vec{E}_x \]

\[ J_y = 0 = nq \mu_n \vec{E}_y + \left( \sigma_n \mu_n r_H \right) \vec{E}_x B_z \]

\[ \vec{E}_y = -\mu_n r_H B_z \vec{E}_x = -\frac{r_H B_z J_x}{nq} \]

\[ \frac{E_y}{J_x B_z} \equiv R_H = \frac{r_H}{(-q)n} \]

\( (R_H: \text{ Hall coefficient}) \)

“measures” the carrier density
(Hall factor, 1 < \( r_H < 2 \) )
example: some numbers

assume “silicon”

\[ n_0 = 10^{16} \text{ cm}^{-3} \]

\[ J_n = 10^2 \text{ A/cm}^2 \]

\[ \mu_n = 1000 \text{ cm}^2/\text{V-s} \]

\[ r_H = 1 \]

\[ B_z = 2,000 \text{ Gauss} \]

\[ (10^4 \text{ Gauss} = 1 \text{ T}) \]

\[ W_y = 1 \mu\text{m} \]

\[ \mathcal{E}_y = -\mu_n r_H B_z \mathcal{E}_x = -\frac{r_H B_z J_x}{n_0 q} \]

\[ \mathcal{E}_y = -1.25 \times 10^2 \text{ V/cm} \]

\[ V_H = -\mathcal{E}_y W_y \approx 13 \text{ mV} \]
outline

1) Magnetoconductivity tensor

2) Resistivity tensor

3) Strong B-fields: Landau levels

4) Shubnikov-DeHaas Oscillations and QHE

5) Summary
B-field dependent DD equation: again

\[ \vec{F}_e = -q\vec{E} - q\vec{v} \times \vec{B} = \frac{d\vec{p}}{dt} \quad \vec{p} = \left(-q\vec{E} - q\vec{v} \times \vec{B}\right) \tau = m^* \vec{v} \]

\[ \vec{v} = -\frac{q\tau}{m^*} \vec{E} - \frac{q\tau}{m^*} \vec{v} \times \vec{B} \]  
(Can be solved exactly for the velocity. See prob. 4.18, Lundstrom.)

\[ \nu_x = -\frac{q\tau}{m^*} E_x - \frac{q\tau}{m^*} \nu_y B_z \]  
2D problem

\[ \nu_y = -\frac{q\tau}{m^*} E_y + \frac{q\tau}{m^*} \nu_x B_z \]  
z-directed B-field
solution

\[ \nu_x = -\frac{q\tau}{m^*} E_x - \frac{q\tau}{m^*} \nu_y B_z \]

\[ \nu_y = -\frac{q\tau}{m^*} E_y + \frac{q\tau}{m^*} \nu_x B_z \]

\[ \omega_c = \frac{qB_z}{m^*} \]

"cyclotron frequency"
solution

\[ \nu_x = \frac{-\mu_n \mathcal{E}_x + \mu_n^2 \mathcal{E}_y B_z}{1 + (\omega_c \tau)^2} \]

Assume \( T = 0 \text{K} \), so there is no need to average the scattering time.

\[ J_x = -nq\nu_x = \frac{\sigma_n}{1 + (\mu_n B_z)^2} (\mathcal{E}_x - \mu_n B_z \mathcal{E}_y) \]

\[ J_y = -nq\nu_y = \frac{\sigma_n}{1 + (\mu_n B_z)^2} (\mathcal{E}_x + \mu_n B_z \mathcal{E}_y) \]

\[ \mu_n B_z = \frac{q \tau}{m^* B_z} = \omega_c \tau \]
magneto-conductivity tensor

\[
\begin{pmatrix}
J_x \\
J_y
\end{pmatrix} = \frac{\sigma_n}{1 + (\mu_n B_z)^2} \begin{pmatrix}
1 & -\mu_n B_z \\
\mu_n B_z & 1
\end{pmatrix} \begin{pmatrix}
\mathcal{E}_x \\
\mathcal{E}_y
\end{pmatrix}
\]

\[
(\omega_c \tau = \mu_n B_z)
\]

\[
J_i = \sigma_{ij} (B_z) \mathcal{E}_j
\]
\[ \sigma_{ij}(B_z) = \frac{nq\mu_n}{1 + \mu_n B_z} \begin{bmatrix} 1 & -\mu_n B_z \\ \mu_n B_z & 1 \end{bmatrix} \]

1) A magnetic field affects both the diagonal and off-diagonal components of the magneto-conductivity tensor.

2) Small magnetic field means: \( \mu_n B_z << 1 \)

\( \omega_c \tau << 1 \)
“Low B-field” means that electrons scatter many times before completing an orbit.

“High B-field” means that electrons can complete an orbit without scattering.
example: some numbers

assume “silicon”

\[ n_0 = 10^{16} \ \text{cm}^{-3} \]
\[ J_n = 10^2 \ \text{A/cm}^2 \]
\[ \mu_n = 1000 \ \text{cm}^2/\text{V-s} \]
\[ r_H = 1 \]
\[ B_z = 2,000 \ \text{Gauss} \]
\[ 10^4 \ \text{Gauss} = 1 \text{T} \]
\[ W_y = 1 \ \mu\text{m} \]

\[ \mu_H B_z \approx 0.02 << 1 \]
\[ \mu_H B_z \approx 10 \rightarrow B_z = 100 \ \text{T} \]

Hall effect measurements with typical laboratory-sized magnets are in the low B-field regime. Except – for very high mobility sample such as modulation doped films.)

Birck Nanotechnology Center: 1-8 T
National High Magnetic Field Lab (Florida State Univ.): 45 T
magneto-resistivity tensor

\[
\begin{align*}
\begin{pmatrix} J_x \\ J_y \end{pmatrix} &= \frac{\sigma_n}{1 + (\mu_n B_z)^2} \begin{pmatrix} 1 & -\mu_n B_z \\ \mu_n B_z & 1 \end{pmatrix} \begin{pmatrix} E_x \\ E_y \end{pmatrix} \\
J_i &= \sigma_{ij} (B_z) E_j
\end{align*}
\]

\[
\begin{align*}
\begin{pmatrix} J_x \\ J_y \end{pmatrix} &= \begin{pmatrix} \sigma_L & -\sigma_T \\ \sigma_T & \sigma_L \end{pmatrix} \begin{pmatrix} E_x \\ E_y \end{pmatrix} \\
\begin{pmatrix} E_x \\ E_y \end{pmatrix} &= \begin{pmatrix} \rho_L & \rho_T \\ -\rho_T & \rho_L \end{pmatrix} \begin{pmatrix} J_x \\ J_y \end{pmatrix}
\end{align*}
\]

\[
\rho_L = \frac{\sigma_L}{\sigma_L^2 + \sigma_T^2} = \frac{1}{\sigma_n}
\]

\[
\rho_T = \frac{\sigma_T}{\sigma_L^2 + \sigma_T^2} = \frac{\mu_n B_z}{\sigma_n}
\]
\[
\begin{align*}
\begin{pmatrix}
J_x \\
J_y
\end{pmatrix}
&= \frac{\sigma_n}{1 + (\mu_n B_z)^2} \begin{bmatrix}
1 & -\mu_n B_z \\
\mu_n B_z & 1
\end{bmatrix}
\begin{pmatrix}
E_x \\
E_y
\end{pmatrix} \\
\begin{pmatrix}
E_x \\
E_y
\end{pmatrix}
&= \frac{1}{\sigma_n} \begin{bmatrix}
1 & \mu_n B_z \\
-\mu_n B_z & 1
\end{bmatrix}
\begin{pmatrix}
J_x \\
J_y
\end{pmatrix}
\end{align*}
\]

1) Longitudinal magneto-resistance is independent of \( B \)
2) Hall voltage is proportional to \( B \)
Hall and integer quantum Hall effect

outline

1) Magnetoconductivity tensor
2) Resistivity tensor
3) **Strong B-fields:** Landau levels
4) Shubnikov-DeHaas Oscillations and QHE
5) Summary
Schrödinger equation with a B-field

\[
\left[ \varepsilon_1 + \frac{(i\hbar \nabla + q\vec{A})}{2m^*} + U(x, y) \right] \Psi(x, y) = E \Psi(x, y)
\]

\[
\vec{B} = \nabla \times \vec{A}
\]

cyclotron frequency

\[ B = B_z \hat{z} \]

\[ \vec{B} = B_z \hat{z} \]

\[ \vec{\nu} \]

\[ k = k_x \hat{x} + k_y \hat{y} \]

\[ \theta \]

\[ \hat{k} \]

\[ \frac{d (\hbar \vec{k})}{dt} = -q \vec{\nu} \times \vec{B} \]

\[ \hbar \frac{dk_x}{dt} = -q \nu_y B_z = \hbar k \frac{d \cos \theta}{dt} = -q \nu \sin \theta B_z \]

\[ \hbar \frac{dk_y}{dt} = +q \nu_x B_z = \hbar k \frac{d \sin \theta}{dt} = +q \nu \cos \theta B_z \]

\[ \frac{d^2 \cos \theta}{dt^2} = -\left( \frac{q \nu B_z}{\hbar k} \right)^2 \cos \theta = -\omega_c^2 \cos \theta \]

\[ \cos \theta (t) = \cos \theta (0) e^{i \omega_c t} \]
cyclootron frequency

\[ \vec{B} = B_z \hat{z} \]

\[ \frac{d (\hbar \vec{k})}{dt} = -q \vec{v} \times \vec{B} \]

harmonic oscillator:

\[ \omega_c = \left( \frac{q \nu B_z}{\hbar k} \right) \]

Quantum mechanically:

\[ E_n = \left( n + \frac{1}{2} \right) \hbar \omega_c \]

“Landau levels”
cyclotron frequency

\[
E_n = \left(n + \frac{1}{2}\right)\hbar \omega_c \quad \omega_c = \left(\frac{q \nu B_z}{\hbar k}\right)
\]

i) parabolic energy bands:

\[
\nu = \frac{\hbar k}{m^*} \quad \omega_c = \frac{q B_z}{m^*}
\]

ii) graphene:

\[
E = \hbar \nu_F k \quad \omega_c = \frac{q B_z}{\left(E/\nu_{F}^2\right)}
\]
effect on DOS

\[ E_n = \left( n + \frac{1}{2} \right) \hbar \omega_c \]

\[ \omega_c = \frac{q B_z}{m^*} \]

\[ D_{2D}(E, B_z) = D_0 \sum_{n=0}^{\infty} \delta \left[ E - \varepsilon_1 - \left( n + \frac{1}{2} \right) \hbar \omega_c \right] \]
The degeneracy of Landau levels is given by the quantum of the magnetic field, $\omega_c = \frac{q B_z}{m^*}$, where $q$ is the charge of the electron, $B_z$ is the magnetic field component in the $z$ direction, and $m^*$ is the effective mass of the electron.

The energy levels, $E_n = \left(n + \frac{1}{2}\right)\hbar \omega_c$, are quantized with $n$ being an integer.

The formula for the degeneracy $D_{2D}(E)$ is:

$$D_0 = \hbar \omega_c \times \frac{m^*}{\pi \hbar^2} = \frac{2qB_z}{\hbar}$$
\[ D_{2D}(E) \]

\[
\Delta E = \frac{m^*}{\pi \hbar^2}
\]

\[ E_n = \left( n + \frac{1}{2} \right) \hbar \omega_c \]

\[ \omega_c = \frac{q B_z}{m^*} \]

\[ D_0 = \frac{2qB_z}{\hbar} \]

\[ \Delta E \Delta t = \hbar \]

\[ \Delta E \approx \hbar / \tau \]

To observe Landau levels: \( \hbar \omega_c >> \Delta E \rightarrow \omega_c \tau >> 1 \)
If $B = 1\, \text{T}$, how many states are there in each LL?

$$D_0 = \frac{2qB_z}{h} = 4.8 \times 10^{10} \, \text{cm}^{-2}$$

If $n_S = 5 \times 10^{11} \, \text{cm}^{-2}$, then 10.4 LL’s are occupied.

How high would the mobility need to be to observe these LL’s?

$$\mu B > 1 \rightarrow \mu > 10,000 \, \text{cm}^2/\text{V-s} \quad (B = 1 \, \text{T})$$

“modulation-doped semiconductors”
variation of LL’s with $B$

1) degeneracy of each level increases with $B$ \[ D_0 = 2\frac{qB_z}{\hbar} \]

2) spacing of levels increases \[ \omega_c = \frac{qB_z}{m^*} \]

(from J.H. Davies, *The Physics of Low-Dimensional Semiconductors*, Cambridge, 1998, Fig. 6.8, p. 226)
modulation doping

$E_F$  
$E_C$  

$E_V$  

wide band gap doped n$^+$

small band gap nominally undoped

1) Magnetoconductivity tensor

2) Resistivity tensor

3) Strong B-fields: Landau levels

4) Shubnikov-DeHaas Oscillations and QHE

5) Summary
SdH oscillations

Longitudinal magneto-resistance

“Shubnikov-deHaas (SdH) oscillations”

For a given sheet carrier density, $n_s$, some (fractional) number of LL's are occupied.
As the $B$-field is varied, the longitudinal resistance will oscillate as the number of filled LL’s changes. The period of the oscillation corresponds to the change in filling factor from one integer to the next.

$$\nu_1 - \nu_2 = 1$$

$$\frac{n_s}{2qB_1/h} - \frac{n_s}{2qB_2/h} = 1$$

$$n_s = \frac{2q}{h} \left( \frac{1}{1/B_1 - 1/B_2} \right)$$
Assume that the Fermi level lies between Landau levels in the bulk, then only the edge states are occupied (edge states play the role of modes).
QHE: quantized Hall Voltage

\[ T = 1 \text{ because } +\nu \text{ and } -\nu \text{ channels are spatially isolated.} \]

(from J.H. Davies, *The Physics of Low-Dimensional Semiconductors*, Cambridge, 1998, Fig. 6.18, p. 240)

\[ V_1 = 0 \text{ and } V_2 = +V \]

\[ V_3 = V_4 = V_1 = 0 \]

\[ V_6 = V_5 = V_2 = +V \]

\[ I = \frac{2q^2}{h} \nu V \]

\[ V_H = V_3 - V_5 = \frac{1}{\nu} \frac{h}{2q^2} I \]

“quantized Hall resistance”
QHE: zero longitudinal magnetoresistance

(from J.H. Davies, *The Physics of Low-Dimensional Semiconductors*, Cambridge, 1998, Fig. 6.18, p. 240)

\[ V_1 = 0 \text{ and } V_2 = +V \]

\[ V_3 = V_4 = V_1 = 0 \]

\[ V_6 = V_5 = V_2 = +V \]

\[ I = \frac{2q^2}{h} \nu V \]

\[ V_x = V_4 - V_3 = 0 \]

zero longitudinal resistance
for a more complete discussion

